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Deforming field theories with $\mathrm{U}(1) \times \mathrm{U}(1)$ global symmetry and their gravity duals

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Abstract: We find the gravity dual of a marginal deformation of $\mathcal{N} = 4$ super Yang Mills, and discuss some of its properties. This deformation is intimately connected with an $\mathrm{SL}(2, \mathbb{R})$ symmetry of the gravity theory. The $\mathrm{SL}(2, \mathbb{R})$ transformation enables us to find the solutions in a simple way. These field theory deformations, sometimes called β deformations, can be viewed as arising from a star product. Our method works for any theory that has a gravity dual with a $\mathrm{U}(1) \times \mathrm{U}(1)$ global symmetry which is realized geometrically. These include the field theories that live on D3 branes at the conifold or other toric singularities, as well as their cascading versions.

Keywords: AdS-CFT and dS-CFT Correspondence, String Duality.

Contents

1. Introduction

The gauge theory/gravity duality (or AdS/CFT) relates field theories to gravitational theories with particular boundary conditions [1−3]. Modifications in the boundary conditions correspond to changes in the field theory lagrangian. In this paper we consider a specific type of deformation of the field theory lagrangian and its corresponding gravity dual. These are sometimes called β deformations. One example of the class of deformations we study is the marginal [4, 5] deformation of $\mathcal{N}=4$ Yang-Mills theory to $\mathcal{N}=1$ which preserves a $\mathrm{U}(1)\times\mathrm{U}(1)$ non-R-symmetry.[1] In general, we consider $\mathrm{U}(N)$ field theories with a

[1]By a non-R-symmetry we mean a symmetry that leaves the $\mathcal{N}=1$ supercharges invariant. In addition, $\mathcal{N}=1$ superconformal theories have a $\mathrm{U}(1)_R$ symmetry.

U(1) $\times$ U(1) global symmetry. The deformation we study can be viewed as arising from a new definition of the product of fields in the lagrangian

$$f * g \equiv e^{i\pi\gamma(Q_f^1 Q_g^2 - Q_f^2 Q_g^1)} fg \tag{1.1}$$

where fg is an ordinary product and (Q^1, Q^2) are the U(1) $\times$ U(1) charges of the fields. Though this prescription is similar in spirit to the one used to define non-commutative field theories [6, 7], the resulting theory is an ordinary field theory. All that happens is that (1.1) introduces some phases in the lagrangian. For example, in the $\mathcal{N} = 4 \to \mathcal{N} = 1$ deformation we mentioned above this results in the change of superpotential

$$\mathrm{Tr}(\Phi_1\Phi_2\Phi_3 - \Phi_1\Phi_3\Phi_2) \to \mathrm{Tr}(e^{i\pi\gamma}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\gamma}\Phi_1\Phi_3\Phi_2)\,. \tag{1.2}$$

We shall call this the "β-deformed" theory. Suppose that we know the gravity dual of the original theory and that this geometry has two isometries associated to the two U(1) global symmetries. Thus the geometry contains a two torus. The gravity description of the deformation (1.1) is surprisingly simple. We just need make the following replacement

$$\tau \equiv B + i\sqrt{g} \longrightarrow \tau_\gamma = \frac{\tau}{1 + \gamma\tau} \tag{1.3}$$

in the original solution, where $\sqrt{g}$ is the volume of the two torus. We can view (1.3) as a solution generating transformation. Namely, we reduce the ten dimensional theory to eight dimensions on the two torus. The eight dimensional gravity theory is invariant under SL$(2,\mathbb{R})$ transformations acting on τ. The deformation (1.3) is one particular element of SL$(2,\mathbb{R})$. This particular element has the interesting property that it produces a non-singular metric if the original metric was non-singular. The SL$(2,\mathbb{R})$ transformation could only produce singularities when $\tau \to 0$. But we see from (1.3) that $\tau_\gamma = \tau + o(\tau^2)$ for small τ. Therefore, near the possible singularities the ten dimensional metric is actually same as the original metric, which was non-singular by assumption.

In the rest of this paper we explain this idea in more detail. Section 2 is devoted to a detailed explanation of the action of the SL$(2,\mathbb{R})$ transformation (1.3) which plays a central role in this paper. We explain how it is associated to (1.1) and point out its intimate connection with non-commutative theories. In section 3 we apply this method to obtain the gravity solution for the marginal deformation of $\mathcal{N} = 4$ super-Yang-Mills. We also discuss various features that arise at rational values of the parameter γ in (1.2). These field theories were studied in [8 − 10, 12, 11, 13 − 17]. In section 4 we discuss more general cases. We discuss a marginal deformation of the conifold theory studied by Klebanov and Witten [18]. We also generalize this to marginal deformations of more general theories based on toric manifolds [20 − 22]. We also point out that this method applies also to non-conformal theories, so it can be applied to obtained a deformed version of the cascading theory of Klebanov and Strassler [23]. I section 5 we present a generalization of this method to theories with three U(1) symmetries, which leads to marginal deformations of $AdS_4 \times S^7$.

In appendix A we give more details on the action of the solution generating transformation and we give the detailed metrics for various cases mentioned in the main text. In appendix B we give present classical string solutions that are associated to BPS states in the β-deformation of $\mathcal{N} = 4$ super Yang-Mills for rational deformation parameters.

2. An $\mathrm{SL}(2,\mathbb{R})$ transformation

Let us consider a string theory background with two U(1) symmetries that are realized geometrically. Namely there are two coordinates φ_1, φ_2 and the two U(1) symmetries act on these two coordinates as shifts of φ_i. Then we will have a two torus parametrized by φ_i, which, in general, will be fibered over an eight dimensional manifold. A simple example is the metric of R^4

$$ds^2 = d\rho_1^2 + d\rho_2^2 + \rho_1^2 d\varphi_1^2 + \rho_2^2 d\varphi_2^2 \,. \tag{2.1}$$

As this example shows, the two torus could contract to zero size at some points but nevertheless the whole manifold is non-singular.

When we compactify a closed string theory on a two torus the resulting eight dimensional theory has an exact $\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$ symmetry which acts on the complex structure of the torus and on the parameter[2]

$$\tau = B_{12} + i\sqrt{g} \tag{2.2}$$

where $\sqrt{g}$ is the volume of the two torus in string metric. The $\mathrm{SL}(2,\mathbb{Z})$ that acts on the complex structure will not play an important role and we forget about it for the moment. At the level of supergravity we have an $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ symmetry. This is not a symmetry of the full string theory. The $\mathrm{SL}(2,\mathbb{R})$ symmetries of supergravity can be used as solution generating transformations. This is a well known trick which was used to generate a variety of solutions in the literature [24]. The $\mathrm{SL}(2,\mathbb{R})$ symmetry that plays a central role in this paper is the one acting as

$$\tau \to \tau' = \frac{\tau}{1+\gamma\tau} \tag{2.3}$$

where τ is given by (2.2). Of course, we can also think of (2.3) as the result of doing a T-duality on one circle, a change of coordinates, followed by another T-duality. When γ is an integer (2.3) is an $\mathrm{SL}(2,\mathbb{Z})$ transformation, but for general γ it is not. This transformation generates a new solution. For example, applying this to (2.1) we get[3]

$$\tau = i\rho_1\rho_2 \to \tau' = \frac{\gamma\rho_1^2\rho_2^2}{1+\gamma^2\rho_1^2\rho_2^2} + i\frac{\rho_1\rho_2}{1+\gamma^2\rho_1^2\rho_2^2} \,. \tag{2.4}$$

The metric after the transformation (2.4) is

$$\begin{aligned} ds^2 &= d\rho_1^2 + d\rho_2^2 + \frac{1}{1+\gamma^2\rho_1^2\rho_2^2}(\rho_1^2 d\varphi_1^2 + \rho_2^2 d\varphi_2^2) \\ B_{12} &= \frac{\gamma\rho_1^2\rho_2^2}{1+\gamma^2\rho_1^2\rho_2^2} \\ e^{2\phi} &= e^{2\phi_0}\frac{1}{1+\gamma^2\rho_1^2\rho_2^2} \end{aligned} \tag{2.5}$$

[2]Throughout this paper we are setting $\alpha' = 1$ and we are using a normalization for the B field such that its period is $B_{12} \sim B_{12} + 1$.

[3]This formula is OK for the bosonic string. For the superstring we should replace $\gamma \to 2\gamma$ if we want an integer gamma to correspond to an $\mathrm{SL}(2,\mathbb{Z})$ symmetry. This is due to the fermion periodicity conditions.

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where the change in the dilaton is due to the fact that the $\mathrm{SL}(2,\mathbb{R})$ transformation leaves the eight dimensional dilaton invariant, not the ten dimensional one. (ϕ_0 is the original ten dimensional dilaton). This type of background is similar to the flux branes [25].[4] It might be possible to quantize strings exactly in these backgrounds, as it was done in [26] for similar cases.

Suppose that we start with a non-singular ten dimensional geometry. After applying (2.3), when is the geometry non-singular?. Let us assume that in the original ten dimensional geometry the B field goes to zero when $\tau_2 \to 0$. This will be obviously true if the original B field is zero, as in the example we considered above, based on (2.1). Note that in order for the original solution to be non-singular we only need that τ_1 goes to an integer when $\tau_2 \to 0$. We might run into trouble if there are different regions where $\tau_2 \to 0$ and τ_1 goes to different integers. In these cases, we produce singularities and our method cannot be applied.[5] Under these assumptions the transformation (2.3) will give us a geometry that is non singular. The reason is that the only points where we could possibly introduce a singularity by performing an $\mathrm{SL}(2,\mathbb{R})$ transformation is where the two torus shrinks to zero size. In this case $\tau_2 \to 0$ and by assumption we also have $\tau_1 \to 0$. If τ is small, then τ' becomes equal to τ and the region near the possible singularity becomes equal to what it was before the transformation. Thus the metric remains non-singular. For initial configurations with non-zero NS or RR fieldstrength the analysis of regularity is a bit longer and is discussed further in appendix A. Notice that (2.3) is the only $\mathrm{SL}(2,\mathbb{R})$ transformation with this property. In a more general situation when the torus is fibered over the remaining eight dimensions a similar argument goes through (see appendix A for more details on the general conditions). This argument also shows that the topology of the solution remains the same. The detailed formulas for the action of the $\mathrm{SL}(2,\mathbb{R})$ transformation in the most general case are given in appendix A.

Let us consider a D-brane on the original background that is invariant under both U(1) symmetries. Such a brane will be left invariant under the action of (2.3). In other words, there is a corresponding brane on the new background. We now ask the question: what is the theory on this brane in the new background? We conjecture the following answer. Suppose that the original brane, on the original background, gave rise to a certain open string field theory. Then the open string field theory on the brane living on the new background is given by changing the start product

$$f *_\gamma g \equiv e^{i\pi\gamma(Q_f^1 Q_g^2 - Q_f^2 Q_g^1)} f *_0 g \tag{2.6}$$

where $*_0$ is the original star product and $Q^i_{f,g}$ are the U(1) charges of the fields f and g. The basic idea leading to this conjecture is the following. In [7] it was pointed out that in the presence of a B field the open string field theory is defined in terms of an open string

[4]In fact if we apply this trick to the initial metric $ds^2 = dr^2 + r^2 d\varphi_1^2 + d\varphi_2^2$ we get a solution where the H_3^{NS} field has a nonzero value at the origin. These solutions were studied in [25].

[5]An example were we get a singular solution arises when we consider an NS five brane and we perform this procedure based on $\mathrm{U}(1) \times \mathrm{U}(1) \in \mathrm{SO}(4)$ acting on the transverse dimensions.

metric and non-commutativity parameter

$$G^{ij}_{\rm close} + \Theta^{ij} = \left(\frac{1}{g+B}\right)^{ij} \sim \frac{1}{\tau} \tag{2.7}$$

where the last expression is schematic. Note that under the transformation (2.3) $1/\tau \to 1/\tau' = 1/\tau + \gamma$. All that happens in (2.7) is that we introduce a non-commutativity parameter $\Theta^{12} = \gamma$. The open string metric remains the same. The reason we called this a "conjecture" rather than a derivation is that [7] considered a constant metric and B field while here we are applying their formulas in a case where these fields vary in spacetime.

Let us now consider branes sitting at the origin. In general, the "origin" is the point where both circles shrink to zero size.[6] Notice that for this brane the transformation (2.6) does *not* lead to a non-commutative field theory at low energies, since the U(1) directions are not along its worldvolume but they are global symmetries of the field theory. The net effect of (2.6) for the field theory living on a brane is to introduce certain phases in the lagrangian according to the rule in (1.1). In other words, starting with the low energy conventional field theory living on the brane, we obtain another conventional field theory with some phases in the lagrangian according to (1.1). Viewing the deformation as a $*$ product allows us to show that all planar diagrams in the new theory are the same as in the old theory [27]. Then, for example, if the original theory was conformal, then this is a marginal deformation to leading order in N. The reader might be bothered by the fact that we are using the arguments in [7] to derive the theory on a brane sitting at a point were $\sqrt{g} = 0$. One indirect argument for our procedure is the following. Start with a $D(p+2)$ brane anti-brane system, wrapped on the two torus, with a magnetic flux of their worldvolume U(1)s on the two torus so that we have net Dp brane charge. These branes can annihilate via tachyon condensation [29] to form the Dp brane at the origin. The brane anti-brane system can be located far from the origin. The process of tachyon condensation is insensitive to the Θ parameter, so it proceeds in the same way in the theory after the $\mathrm{SL}(2,\mathbb{R})$ transformation as in the theory before the transformation. The net result is that we obtain the same field theory on the Dp brane at the origin, but with the extra phases (1.1).

Notice that these arguments are completely general and can also be applied for toric singularities, such as the conifold. In those cases we use the U(1) symmetries of the toric manifold and apply (2.3). The field theories on D-branes living at the singularity are deformed according to (1.1).

The description in terms of this $\mathrm{SL}(2,\mathbb{R})$ transformation (2.3) and its associated modification of the theory (2.6) unifies the conceptual description of non-commutative field theories [6, 7] and dipole theories [28] with the so called β deformations of field theories, which are the main subject of this paper.

[6] If none of the circles shrinks, then we have the non-commutative theories considered in [7]. If one circle shrinks and the other does not shrink then we are lead to the dipole theories studied in [28]. These arise after taking the metric $ds^2 = d\rho^2 + \rho^2 d\varphi_1^2 + d\phi_2^2$, putting a D-brane at the origin which is extended in φ_2 and performing the transformation (2.3).

There are some special features that occur when $\gamma = m/n$, with m, n coprime. In this case it is possible to do a further SL(2, $\mathbb{Z}$) transformation on τ' to give

$$\tau'' = \frac{a\tau' + b}{c\tau' + d} = \frac{a\tau' + b}{-m\tau' + n} = \frac{1}{n^2}\tau + \frac{b}{n} \tag{2.8}$$

$$\text{with } an + mb = 1\,. \tag{2.9}$$

Since m, n are coprime, there is a solution to (2.9). The final expression for τ'' (2.8) is precisely that of a $Z_n \times Z_n$ orbifold with discrete torsion of the original torus. The discrete torsion is simply the last term in (2.8). Up to SL(2, $\mathbb{Z}$) transformations the final form of τ'' is independent of our choice of b. The fact that D-branes in orbifolds with discrete torsion are related to deformations of the field theory of the form (1.1) was pointed out in [30] and further explored in [8].

An interesting question is whether the deformation that we are doing preserves supersymmetry. In principle we can perform this transformation independently of whether we break or preserve supersymmetry, but sometimes we are interested in the ones that preserve it. If the original ten dimensional background is supersymmetric under a supersymmetry that is invariant under U(1) $\times$ U(1), then the deformed background will also be invariant under this supersymmetry. As an example, let us start with $R^4 \times R^6$. In R^6 we choose complex coordinates z^i. Then we choose a subgroup $\mathrm{U}(1)\times\mathrm{U}(1) \subset \mathrm{SU}(3) \subset \mathrm{SO}(6)$. Since these U(1) symmetries act geometrically, we can apply our construction. The resulting solution preserves $\mathcal{N} = 2$ supersymmetry in four dimensions. The interested reader can find its explicit form in appendix A. A D3-brane at the origin leads to an $\mathcal{N} = 1$ theory. Actually, we obtain the marginal deformation (1.2). We discuss this case more explicitly in the next section. More generally we can imagine a toric singularity with three U(1) symmetries. We can pick two U(1) symmetries that — leave the spinor invariant. Then the transformation (2.3) will deform the theory that lives at singularity. This same transformation can give us the near horizon geometry of the new theory.

There is no reason we should restrict to conformal field theories, we can do this type of deformation in non-conformal field theories. The simplest examples would be the theories living on D-p branes in flat space [31]. More complicated examples include the cascading theory considered by Klebanov-Strassler [23].

So far we have been discussing the theory on the brane that arises after putting branes in backgrounds where we have performed our SL(2, $\mathbb{R}$) transformation. We can as easily consider the gravity duals of these theories. If we know the gravity dual of the field theory living on a D-brane in the original background, then the gravity dual of the deformed field theory living on the D-brane on the new background is given by performing the SL(2, $\mathbb{R}$) transformation (2.3) on the original solution. We discuss explicit examples in the following sections.

What we have discussed so far applies both for IIA and IIB string theory. In fact, some of our discussion also applies for bosonic string theory. Let us now concentrate in the case of IIB supergravity. This theory has an SL(2, $\mathbb{R}$)$_s$ symmetry already in ten dimensions.[7]

[7]The subindex s in SL(2, $\mathbb{R}$)$_s$ was introduced to avoid confusion with other SL(2, $\mathbb{R}$) symmetries of the problem.

Once we compactify the theory on a two torus, this $SL(2,\mathbb{R})_s$, together with the $SL(2,\mathbb{R})$ symmetry that acts on (2.2) form an $SL(3,\mathbb{R})$ group. These can be used to generate more general solutions. Perhaps a simple way to start is to first do an S duality, then the transformation (2.3), and finally an S-duality again. This combined transformation acts on a tau parameter that involves the RR field. Actually, a compactification of type-IIB supergravity on a two torus has $SL(3,\mathbb{R})\times SL(2,\mathbb{R})$ symmetry.[8] The second $SL(2,\mathbb{R})$ is the one acting on the complex structure of the two torus and will not play an important role now. Starting with a general non-singular ten dimensional solution of type-IIB supergravity with two U(1) geometric symmetries we can act with $SL(3,\mathbb{R})$ transformations to generate new solutions. If we are interested in generating non-singular solutions, starting from solutions where the two torus goes to zero in some regions, then the SL(3) matrix cannot be arbitrary. It to be of the form

$$h_3 = \begin{pmatrix} 1 & 0 & 0 \\ \gamma & a & b \\ \sigma & c & d \end{pmatrix}, \qquad \text{where } h_2 \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \in SL(2,\mathbb{R})\,. \tag{2.10}$$

In total, such solutions contain four parameters. Two of them arise from the axion-dilaton τ_s which parametrizes $SL(2,\mathbb{R})/\,U(1)$ associated to the matrix h_2.[9] The other two parameters are γ and σ. The latter are periodic variables $\gamma \sim \gamma+1,\ \sigma\sim\sigma+1$ and they form a representation of $SL(2,\mathbb{Z})_s$ which acts on τ_s.

If we have more U(1) symmetries we can do other deformations. For example, let us consider an eleven dimensional supergravity solution with three U(1) symmetries. This solution contains a three torus. The dimensional reduction of eleven dimensional supergravity on a T^3 is the same as IIB on the T^2 and has an $SL(3,\mathbb{R})\times SL(2,\mathbb{R})$. We can do an $SL(2,\mathbb{R})$ transformation on

$$\tau = C_{123} + i\sqrt{g} \tag{2.11}$$

where $\sqrt{g}$ is the volume of the three torus. We discuss this a bit further in section 5.

3. Marginal deformations of $\mathcal{N}=4$ Super Yang Mills

3.1 Field theory

As studied in [4, 5], there is a three parameter family of marginal deformations of $\mathcal{N}=4$ Super Yang Mills that preserve $\mathcal{N}=1$ supersymmetry. These theories have a superpotential of the form

$$h\ \mathrm{Tr}(e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2) + h'\,\mathrm{Tr}(\Phi_1^3+\Phi_2^3+\Phi_3^3) \tag{3.1}$$

where Φ_i are the three chiral superfields. The parameters h, h', β are complex. In addition we have the gauge coupling. The condition of conformal invariance imposes only one

[8] The $SL(3,\mathbb{R})$ symmetry is most clearly seen by viewing this as a compactification of M-theory on T^3.

[9] Do not confuse τ_s, which parametrizes the eight dimensional axion dilaton, with τ defined in (2.2).

condition among these four parameters [5]. In this paper we set $h' = 0$. In this case, besides the $\mathrm{U}(1)_R$ symmetry, we have a $\mathrm{U}(1) \times \mathrm{U}(1)$ global symmetry generated by

$$\begin{aligned} \mathrm{U}(1)_1: \ & (\Phi_1, \Phi_2, \Phi_3) \to (\Phi_1, e^{i\varphi_1}\Phi_2, e^{-i\varphi_1}\Phi_3) \\ \mathrm{U}(1)_2: \ & (\Phi_1, \Phi_2, \Phi_3) \to (e^{-i\varphi_2}\Phi_1, e^{i\varphi_2}\Phi_2, \Phi_3)\,. \end{aligned} \tag{3.2}$$

This symmetry leaves the superpotential invariant. It also leaves the supercharges invariant. The Leigh-Strassler argument says that we have a 2 dimensional manifold of $\mathcal{N} = 1$ CFTs with $\mathrm{U}(1) \times \mathrm{U}(1)$ global symmetry (beyond the usual $\mathrm{U}(1)_R$ symmetry). This field theory been studied previously in [8 – 10, 12, 11, 13 – 17]. Note that if we start with $\mathrm{U}(N)$ theory the fields $\mathrm{Tr}[\Phi_i]$ couple through the superpotential (3.1) to the $\mathrm{SU}(N)$ fields. It was noted in [16] that these couplings flow to zero in the IR. So in the IR we have an $\mathrm{SU}(N)$ theory.

We will now review a few aspects of this field theory. The theory is invariant under a discrete Z_3 symmetry which acts as a cyclic permutation of the three chiral fields. The physics turns out to be periodic in the variable β. In fact, we can think of the variable β as living on a torus with complex structure τ_s, where τ_s is related to gauge coupling and theta parameter of the field theory [11]. The theory has an $\mathrm{SL}(2, \mathbb{Z})$ duality group. The variable τ_s was chosen in [11] so that it transforms in the usual way under S-duality. Then β transforms as a modular form [11] (see also [32]). In other words

$$\tau_s \to \frac{a\tau_s + b}{c\tau_s + d}\,, \qquad \beta \to \frac{\beta}{c\tau_s + d}\,, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})\,. \tag{3.3}$$

The periodicity in β is

$$\beta \sim \beta + 1 \sim \beta + \tau_s\,. \tag{3.4}$$

The first identification is obvious from the superpotential (3.1). The second is not clear perturbatively, but it follows from S-duality. It is also useful to parametrize β as

$$\beta = \gamma - \tau_s \sigma \tag{3.5}$$

where γ and σ are real variables with period one. Note that the description of the β deformation as a star product, as in (1.1) is valid for real β. For complex β we are just complexifying this parameter in the superpotential, which is not the same as using the prescription (1.1) with complex β to construct the component lagrangian.

Let us find the Coulomb branch. A general analysis of the Coulomb branch of this theory can be found in [14, 15]. The F-term constraints can be written as

$$\Phi_2\Phi_3 = q\Phi_3\Phi_2\,, \qquad \Phi_1\Phi_2 = q\Phi_2\Phi_1\,, \qquad \Phi_3\Phi_1 = q\Phi_1\Phi_3\,, \qquad q = e^{2\pi i\beta}\,. \tag{3.6}$$

For generic β the Coulomb branch consists of diagonal matrices where in each entry two of the Φ_i are zero and only one is non-zero. So we can have, for example $\Phi_1 \neq 0$ and $\Phi_2 = \Phi_2 = 0$. For $\sigma = 0$ and a rational γ we have new features, there are additional regions in the Coulomb branch where the matrices form the fuzzy torus algebra. For $\gamma = m/n$ we can solve (3.6) with [8]

$$\Phi_1 = xU^m\,, \qquad \Phi_2 = yV\,, \qquad \Phi_3 = zV^{-1}U^{-m}\,, \qquad UV = e^{i2\pi/n}VU \tag{3.7}$$

with U and V the standard matrices of the non-commutative torus[10] and x, y, z are complex numbers. We can think of these solutions as D3 branes forming toroidal D5 branes via the Myers effect [33].

S-duality implies that there should be such additional branches whenever the complex variable β is a rational point on the torus. In other words, whenever there is an integer n such that the point $n\beta$ is a point on the lattice generated by $1, \tau_s$. If

$$n(\gamma, \sigma) = (p, q)\,, \qquad n, p, q \in Z \tag{3.8}$$

we expect that n D3 branes can form a toroidal (p, q) fivebrane. The appearance of this branch is not clear from perturbation theory, but was explored via matrix model techniques [34] in [14, 15]. Of course, we can only have these branches in theories where $N \geq n$.

In addition, the SU(N) theory has another branch,[11] even for generic β. This branch arises because in SU(N) we have to project the equations (3.6) into their traceless part. So we can have solutions where, for example, the matrices Φ_i are diagonal and such that $(\Phi_i)_{ll}(\Phi_j)_{ll}$ (no sum) is a constant independent of the matrix index l. In this paper, we will not describe this branch from the supergravity side.

It is also interesting to consider the chiral primary operators since they correspond to BPS states on the gravity side. For generic β there is a single single trace BPS operator with each of the following charges [8, 9]

$$(J_1, J_2, J_3) = (k, 0, 0)\,, \qquad (0, k, 0)\,, \qquad (0, 0, k)\,, \qquad (k, k, k) \tag{3.9}$$

where k is an arbitrary integer. In addition we can have multitrace operators formed by products of single trace operators. For $\sigma = 0$ and $\gamma = m/n$ (m, n coprime) there are more single trace operators with

$$(J_1, J_2, J_3) = (k_1, k_2, k_3)\,, \qquad k_1 = k_2 = k_3 \bmod n\,. \tag{3.10}$$

We expect that the chiral ring has special features for more generic rational β (3.8), but we are not aware of a discussion of this point in the literature.

3.2 Supergravity solution

It follows from the analysis of the Kaluza Klein spectrum on $AdS_5 \times S^5$ that there is a massless field in AdS_5 that corresponds to the deformation in question. In fact, there are more massless fields in AdS_5 than there are exactly marginal deformations. In [35] the super-gravity equations were analyzed to second order and a constraint was found. There are as many solutions to this constraint as there are exactly marginal deformations of $\mathcal{N} = 4$.[12]

[10] $U = diag(1, e^{i2\pi/n}, \cdots, e^{i2\pi(n-1)/n})$, and V has non-vanishing elements $V_{i+1,i} = V_{1,n} = 1$.

[11] This can be obtained from a massless limit of the one discussed in [11].

[12] Singular solutions were found in [36]. They are not the gravity duals of the β-deformations, which are non-singular.

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In this section we will show how to get the exact solution for deformations which preserve U(1) × U(1) global symmetry. All we need to do is to apply the method described in the previous section. Let us start by writing the metric of S^5 in the form[13]

$$
\begin{aligned}
\frac{ds^2}{R^2} &= \sum_{i=1}^{3} d\mu_i^2 + \mu_i^2 d\phi_i^2 \,, \qquad \text{with} \sum_i \mu_i^2 = 1 && (3.11)\\
\frac{s^2}{R^2} &= d\alpha^2 + s_\alpha^2 d\theta^2 + c_\alpha^2 (d\psi - d\varphi_2)^2 + s_\alpha^2 c_\theta^2 (d\psi + d\varphi_1 + d\varphi_2)^2 + s_\alpha^2 s_\theta^2 (d\psi - d\varphi_1)^2 \\
&= d\alpha^2 + s_\alpha^2 d\theta^2 + \frac{9 c_\alpha^2 s_\alpha^2 s_{2\theta}^2}{4c_\alpha^2 + s_\alpha^2 s_{2\theta}^2} d\psi^2 + s_\alpha^2 \left[d\varphi_1 + c_\theta^2 d\varphi_2 + c_{2\theta} d\psi\right]^2 + \\
&\quad + (c_\alpha^2 + s_\alpha^2 s_\theta^2 c_\theta^2) \left[d\varphi_2 + \frac{(-c_\alpha^2 + 2 s_\alpha^2 s_\theta^2 c_\theta^2)}{c_\alpha^2 + s_\alpha^2 s_\theta^2 c_\theta^2} d\psi \right]^2 . && (3.12)
\end{aligned}
$$

Notice that the two U(1) symmetries (3.2) act by shifting φ_1, φ_2. So the two torus we were talking about in our general discussion has a metric given by the last line in (3.12). Actually, we can compute the τ parameter of this two torus

$$
\tau = i\sqrt{g_0} = i[R^2 s_\alpha^2 (c_\alpha^2 + s_\alpha^2 s_\theta^2 c_\theta^2)]^{1/2} = iR(\mu_1^2\mu_2^2 + \mu_2^2\mu_3^2 + \mu_1^2\mu_3^2)^{1/2} \qquad (3.13)
$$

where $R = (4\pi g_s N)^{1/4}$. Note that the terms involving ψ in the last line (3.12) become gauge fields in the eight dimensional theory. After we apply the transformation (2.3) we can find the solution corresponding to the gravity dual of the deformed theory

$$
\begin{aligned}
ds^2_{str} &= R^2 \left[ds^2_{AdS_5} + \sum_i (d\mu_i^2 + G\mu_i^2 d\phi_i^2) + \hat{\gamma}^2 G \mu_1^2\mu_2^2\mu_3^2 (\sum_i d\phi_i)^2 \right] && (3.14)\\
G^{-1} &= 1 + \hat{\gamma}^2 (\mu_1^2\mu_2^2 + \mu_2^2\mu_3^2 + \mu_1^2\mu_3^2)\,, \qquad \hat{\gamma} = R^2\gamma \,, \qquad R^4 \equiv 4\pi e^{\phi_0} N && (3.15)\\
e^{2\phi} &= e^{2\phi_0} G \\
B^{NS} &= \hat{\gamma} R^2 G (\mu_1^2\mu_2^2 d\phi_1 d\phi_2 + \mu_2^2\mu_3^2 d\phi_2 d\phi_3 + \mu_3^2\mu_1^2 d\phi_3 d\phi_1) \\
C_2 &= -3\gamma (16\pi N) w_1 d\psi \,, \qquad \text{with } dw_1 = c_\alpha s_\alpha^3 s_\theta c_\theta d\alpha d\theta \\
C_4 &= (16\pi N)(w_4 + G w_1 d\phi_1 d\phi_2 d\phi_3) \\
F_5 &= (16\pi N)(\omega_{AdS_5} + G\omega_{S^5})\,, \qquad \omega_{S^5} = dw_1 d\phi_1 d\phi_2 d\phi_3 \,, \qquad \omega_{AdS_5} = dw_4
\end{aligned}
$$

where ω_{S^5} is the volume element of a unit radius S^5. The metric is written in string frame. We have made manifest the Z_3 symmetry which is broken in the intermediate stages when we write the metric as in (3.12).

Let us first examine the regime of validity of this solution. The solution which is presented here has small curvature as long as

$$
R\gamma \ll 1 \,, \qquad R \gg 1 \,. \qquad (3.16)
$$

The first inequality can be understood as the condition that (at a generic point) the two torus does not become smaller than the string scale after the transformation. We also

[13] To save space we define $s_\alpha = \sin\alpha$, $c_\alpha = \cos\alpha$, etc.

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computed the square of the Riemann tensor on the deformed fivesphere and looked at the region where it is a maximum as a check of the first condition (3.16).

Suppose that $\gamma = 1/n$. For these cases the general argument presented in (2.8) shows that the solution is T-dual (or, more precisely, $SL(2,\mathbb{Z})$ equivalent) to a $Z_n \times Z_n$ orbifold with discrete torsion [30, 8]. Then we see from (3.16) that the solution presented above (3.14) has high curvature if n is a relatively small number, while the orbifold description will be weakly coupled. On the other hand, if

$$n \gg R \sim (g_{YM}^2 N)^{\frac{1}{4}} \tag{3.17}$$

our solution will be weakly curved. But in this case the orbifold description would contain circles smaller than the string scale and would not be a good description. If wanted to check that γ is a periodic variable, $\gamma \sim \gamma + 1$, we have to go trough a region where we do not trust the gravity solution. Nevertheless, by construction, our solution formally has this periodicity, after doing the appropriate duality ($SL(2,\mathbb{Z})$ transformation). The topology of this solution (3.14) is always $AdS_5 \times S^5$, since our transformation (2.3) does not change the topology.

3.3 Coulomb branches and chiral primaries

We now explore the special Coulomb branches that appear for rational γ. Perhaps a simple way to understand the emergence of these branches is the following. The eight dimensional theory that we obtain after compactification on the two torus has an SL(3) symmetry. The original D3 branes, together with the D5 and NS5 wrapped on the two torus transform under the fundamental representation of SL(3). In other words we have the transformation law

$$(N_{D3}, N_{D5}, N_{NS5}) \to (N_{D3}, N_{D5} + \gamma N_{D3}, N_{NS5}) \tag{3.18}$$

under (2.3). For non-zero σ we will also shift the NS5-brane charge. So, suppose that we start from some number of D3 branes in the Coulomb branch, at some point where the two torus has finite size. Then, after applying (2.3) we see that the D5 charge will be proportional to γN_{D3}. In general this does not obey the D5 charge quantization condition and is not an allowed brane. But for the special values $\gamma = m/n$ and N_{D3} a multiple of n, then we do obey the D5 charge quantization condition. Since the original configuration was BPS, this new configuration will also be BPS.

Just to check our formulas, let us look at this branch more explicitly in the probe approximation. Notice that by taking first the limit $N \to \infty$, keeping n and R fixed such that (3.16) (3.17) are obeyed, we can ensure that the probe approximation is valid. Let us write down the Dirac-Born-Infeld action for the D5 branes, and let us set the RR scalar to zero for simplicity.

$$S = \int e^{-\phi}\sqrt{det(g + F - B)} - \int C_6 + (F - B) \wedge C_4 \,. \tag{3.19}$$

We will now show that $C_6 - B \wedge C_4$ is zero. C_6 is determined by the equation[14]

$$dC_6 = *dC_2 + H_3 \wedge C_4 \,, \qquad H = dB \tag{3.20}$$

So we find that the particular combination coupling to the D5 brane obeys the equation

$$d(C_6 - B \wedge C_4) = *dC_2 - B \wedge dC_4 = *dC_2 - B \wedge F_5 \tag{3.21}$$

where we used that in our background $(B)^2 dC_2 = 0$. It can be checked that the two terms in the right hand side of (3.21) cancel each other.[15] Since (3.21) is zero, the coupling of the 5-brane to $C_6 - B \wedge C_4$ is constant. We can go to a region where the D5 shrinks to a D3 brane where this coupling is absent. So we conclude that the constant in question is zero. Thus, we find that the DBI action for a D5 brane wrapping the two torus and extended along a Poincare slice of AdS_5 is

$$S \sim r^4 N_5 \left[e^{-\phi} \sqrt{G^2 g_0 + (F_{12} - B_{12})^2} - F_{12} \right] \tag{3.22}$$

where the factor of r^4 comes from the AdS part, N_5 is the number of D5 branes, g_0 is the determinant of the metric of the original two torus (3.13) and G is the function in (3.15). If we pick $F_{12} = 1/\gamma$, then the action (3.22) vanishes, due to the form of the B field and dilaton in (2.3). Of course, in order to obey the quantization condition for D3 brane charge (or the U(1) flux on the D5 brane) we need that $N_5 \gamma^{-1}$ is an integer. This can be obeyed if $\gamma = m/n$ and $N_5 = m$. This branch corresponds to the non-commutative branch described in (3.7).

Suppose that we sit at a point in the Coulomb branch with a large number of coincident D5 branes so that we cannot ignore their backreaction on the geometry. We can find the resulting gravity solution in the following way.[16] We start from a solution dual to $\mathcal{N} = 4$ SYM on the Coulomb branch. We choose this solution so that it contains D3 branes smeared on the two torus appearing in our discussion. Then we perform the SL$(2, \mathbb{R})$ transformation (2.3). This gives us a solution which is regular everywhere except on the two-tori where we smeared the original D3 branes. In the vicinity of the two torus the metric looks similar to that of D5 branes on a non-commutative two torus.[17] But extremely close to the two torus, one would need to perform T-dualities, similar to those described in [31] which would give us $AdS_5 \times S^5$ in the extreme IR. The S-dual of this configuration was explored in [13] as a deconstruction of the NS5 brane theory on a two torus.

[14] See [37], for example. To write these equations collect all RR fields into a single form $C = C_0 + C_2 + \cdots$. Then the field strength $G = dC - H \wedge C$ is gauge invariant under $C \to C + d\Lambda - H \wedge \Lambda$ and self dual $G = *G$. The gauge invariant coupling to D-branes is $\int e^{F-B} C$.

[15] The equations of motion for C_2 ensure that the exterior derivative of the right hand side is zero. It turns out that the two terms in the right hand side of (3.21) have the same functional form, so that they could differ just by a numerical coefficient. It is clear that the numerical coefficients should be such that the two last terms in (3.21) cancel precisely. Otherwise, its exterior derivative would be non-zero.

[16] These metrics were described in an approximate way in [13].

[17] In fact, the gravity solutions for non-commutative field theories [38, 39] can be obtained by applying the SL$(2, \mathbb{R})$ transformation (2.3) to two of the worldvolume directions of the gravity solution dual to the ordinary field theory.

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Let us analyze the chiral primary states. Let us first view the chiral primaries as classical string solutions. This will be a good description when their charges are large. Then we see that pointlike strings, or particles, with momenta

$$(J_1, J_2, J_3) = (J, 0, 0)\,, \qquad (0, J, 0)\,, \qquad (0, 0, J)\,, \qquad (J, J, J) \tag{3.23}$$

lead to BPS states for any γ. This agrees with the field theory analysis.

Something more interesting happens at $\gamma = m/n$. In this case we can get BPS solutions which contain strings wrapped on the cycles of the two torus on which we are doing the $\mathrm{SL}(2,\mathbb{R})$ transformation. These are contractible cycles in the full geometry, so this winding is topologically trivial. Nevertheless, this implies that the corresponding string states are macroscopic strings and are not pointlike. The resulting string states are rather similar to those analyzed by [40]. Of course these are solutions to the equations of motion for the string. Again, the easiest way to find the solutions is to formally do the $\mathrm{SL}(2,\mathbb{R})$ transformation starting from a generic BPS pointlike string in the original background. This state will have momenta n_1, n_2 on the two circles of the two torus. Formally, after the transformation (2.3) the momenta are the same and the winding numbers are $w_1 = -\gamma n_2, \;\; w_2 = \gamma n_1$ Again, when $\gamma = m/n$ the quantization for winding is obeyed if $n_1, \; n_2$ are multiples of n. This condition has to be obeyed if we are considering a classical string solution. The fact that n_1, n_2 are multiples of n implies that $J_1 = J_2 = J_3 \bmod(n)$, as in (3.10). In appendix B we present the explicit classical solutions.

It should be noted that some of the general solutions for BPS states presented in [41] can be used to generate similar solutions in our case. Namely, if we start from a solution in [41] representing a half BPS state that corresponds to a fermion droplet that is circularly symmetric, then the ten dimensional solution will have the two necessary isometries so that we can apply our $\mathrm{SL}(2,\mathbb{R})$ transformation (2.3). In this way we generate gravity solutions corresponding to some of the BPS states of the β-deformed theory.

3.4 General deformation and S-duality

So far we have been discussing the case when $\sigma = 0$, we can easily generate the solutions for non-zero σ by performing $\mathrm{SL}(2,\mathbb{R})_s$ transformations of the solutions with $\sigma = 0$. Here we are referring to the $\mathrm{SL}(2,\mathbb{R})_s$ symmetry group of the ten dimensional theory, which should not be confused with the $\mathrm{SL}(2,\mathbb{R})$ group that we used in (2.3). In other words, start with $AdS_5 \times S^5$ and we perform a more general $\mathrm{SL}(3,\mathbb{R})$ transformation in the eight dimensional theory. The resulting solution is the following.

$$ds_E^2 = R_E^2 G^{-1/4}\left[ds^2_{AdS_5} + \sum_i (d\mu_i^2 + G\mu_i^2 d\phi_i^2) + \frac{|\gamma - \tau_s\sigma|^2}{\tau_{2s}} R_E^4 \mu_1^2\mu_2^2\mu_3^2\Big(\sum_i d\phi_i\Big)^2\right] \tag{3.24}$$

$$e^{-\phi} = \tau_{2s} G^{-1/2} H^{-1}\,, \qquad \chi = \tau_{2s}\sigma(\gamma - \tau_{1s}\sigma) H^{-1} g_{0,E} + \tau_{1s} \tag{3.25}$$

$$G^{-1} \equiv 1 + \frac{|\gamma - \tau_s\sigma|^2}{\tau_{2s}} g_{0,E}\,, \qquad H \equiv 1 + \tau_{2s}\sigma^2 g_{0,E}\,, \qquad \tau_s = \tau_{1s} + i\tau_{2s} \tag{3.26}$$

$$g_{0,E} = R_E^4(\mu_1^2\mu_2^2 + \mu_2^2\mu_3^2 + \mu_1^2\mu_3^2)\,, \qquad R_E^4 = 4\pi N \tag{3.27}$$

$$B^{NS} = \frac{\gamma - \tau_{1s}\sigma}{\tau_{2s}} R_E^4 G w_2 - \sigma 12 R_E^4 w_1 d\psi \tag{3.28}$$

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$$
\begin{aligned}
C_2 &= [-\tau_{2s}\sigma + \frac{\tau_{1s}}{\tau_{2s}}(\gamma - \tau_{1s}\sigma)]R_E^4 G w_2 - \gamma 12 R_E^4 w_1 d\psi \\
dw_1 &= c_\alpha s_\alpha^3 s_\theta c_\theta d\alpha d\theta\,, \qquad w_2 = (\mu_1^2\mu_2^2 d\phi_1 d\phi_2 + \mu_2^2\mu_3^2 d\phi_2 d\phi_3 + \mu_3^2\mu_1^2 d\phi_3 d\phi_1) \\
F_5 &= 4R_E^4(\omega_{AdS_5} + G\omega_{S^5})\,, \qquad \omega_{S^5} = dw_1 d\phi_1 d\phi_2 d\phi_3
\end{aligned} \tag{3.29}
$$

where we wrote the Einstein metric. This metric depends on the complex parameter τ_s, which we identify with the τ_s of the field theory and also on γ, σ which we identify with the same parameters of the gauge theory (3.5). We can check that the solution transforms appropriately under S-duality transformations (3.3).

An interesting question to ask is: What is the metric in the space of couplings?. In other words, what is the Zamolodchikov metric for these marginal deformations. In the supergravity regime this metric can be computed from our solution. It is simplest to think about this problem from the eight dimensional point of view, after reducing on the two torus. The eight dimensional field theory has an $\mathrm{SL}(3,\mathbb{R})$ invariance (plus another $\mathrm{SL}(2,\mathbb{R})$ symmetry that is not important now) and it has scalar fields living on the $\mathrm{SL}(3,\mathbb{R})/\,\mathrm{SO}(3)$ coset. The fields τ_s, γ, σ parametrize some of the fields in this coset. The fifth field is a scalar that measures the size of the two torus in the IIB language. We are not free to change it if we want to have a solution that is non-singular in ten dimensions. $\mathrm{SL}(3,\mathbb{R})$ invariance then determines the action in eight dimensions, which in turn gives us the five dimensional action on AdS_5 for the massless scalar fields in AdS_5 which are dual to the marginal deformations. The bulk action is

$$
S = \frac{N^2}{16\pi^2}\int_{AdS_5}\left[\frac{\partial\tau_s\partial\bar\tau_s}{\tau_{2\,s}^2} + C\frac{|\partial\gamma - \tau_s\partial\sigma|^2}{\tau_{2\,s}}\right] \tag{3.30}
$$

where the integral is over an AdS_5 space of radius one and C is a constant. The overall coefficient is the same as the one appearing in the computation in [42], which determines the coefficient for the dilaton. This result (3.30) is derived as follows. We first write the action in eight dimensions. It is convenient to parametrize an element of $\mathrm{SL}(3)/\,\mathrm{SO}(3)$ as

$$
l = \begin{pmatrix} 1 & 0 & 0 \\ \gamma & \sqrt{\tau_{2s}} & \frac{\tau_{1s}}{\sqrt{\tau_{2\,s}}} \\ \sigma & 0 & \frac{1}{\sqrt{\tau_{2s}}} \end{pmatrix}\begin{pmatrix} g^{1/3} & 0 & 0 \\ 0 & g^{-1/6} & 0 \\ 0 & 0 & g^{-1/6} \end{pmatrix}. \tag{3.31}
$$

Here g is the determinant of the metric on the internal two torus. We can then compute the metric by computing the matrix $M = ll^t$ which contains only the gauge invariant information on the coset. The parametrization (3.31) amounts to a choice of gauge. The metric in the gravity solution is determined as

$$
\mathrm{Tr}[\partial M \partial M^{-1}]\,. \tag{3.32}
$$

We see that this metric is $\mathrm{SL}(3,\mathbb{R})$ invariant. Computing it for (3.31) we find a metric like (3.30) were $C \to g_{0,E}$ where $g_{0,E}$ is defined in (3.27) and is a function of some of the angles of the sphere. Finally we conclude that C in (3.30) is

$$
C = \langle g_{0,E}\rangle = R_E^4\langle\mu_1^2\mu_2^2 + \mu_2^2\mu_3^2 + \mu_1^2\mu_3^2\rangle = \pi N \tag{3.33}
$$

where the average is over the five sphere. This average arises because we are interested in the five dimensional action on AdS_5. We see that the N and g_{YM} dependence match the weak coupling result. We have not checked explicitly the overall numerical coefficient,[18] but the results of [45] imply that the weak coupling result will be the same as the gravity result. The authors of [45] considered two point functions of operators associated to higher Kaluza-Klein harmonics of the dilaton. The first Kaluza-Klein harmonic of the dilaton is related by $\mathcal{N}=4$ supersymmetry to the operator corresponding to a small change in γ.

The discussion about the Coulomb branch is very similar to what we had before. Whenever we are at a rational point on the two torus (3.8) we can have a (p,q) fivebrane in the bulk describing a new piece of the Coulomb branch. Similarly, there are BPS chiral primary states associated to (p,q) strings wrapping the two torus.

3.5 pp-wave limit

Starting from the general solution (3.24) we can take interesting pp-wave limits. For example, we can start with a maximum circle on the deformed S^5 at $\mu_1 = 1$. Lightlike trajectories along this circle, which remain at the origin in AdS_5, correspond to BPS operators of the form $\mathrm{Tr}[\Phi_1^J]$. We can now take a pp-wave limit

$$\begin{aligned} &J, R \to \infty\,, \quad \text{with } -p_- = \frac{J_1}{R^2}\,, \quad \tilde\gamma \equiv (\gamma - \tau_{1\,s}\sigma)R^2\,, \quad \tilde\sigma \equiv \tau_{2\,s}\sigma R^2\,, \quad \text{fixed} \\ &x^- = (t-\phi^1)R^2\,, \quad x^+ = t\,, \quad \text{fixed} \end{aligned} \tag{3.34}$$

with the rest of the coordinates in AdS_5 and (the deformed) S^5 scaling in the same was as the usual pp-wave limit [43, 44].

The metric then becomes

$$ds^2 = -2dx^+dx^- - [r^2 + (1+\tilde\gamma^2+\tilde\sigma^2)y^2](dx^+)^2 + d\vec y^{\,2} + d\vec r^{\,2} \tag{3.35}$$

$$H^{NS} = -2\tilde\gamma dx^+(dy_1dy_2 - dy_3dy_4) \tag{3.36}$$

$$H^{RR} - \tau_{1\,s}H^{NS} = 2\tau_{2\,s}\tilde\sigma dx^+(dy_1dy_2 - dy_3dy_4) \tag{3.37}$$

$$F_5 = \tau_{2\,s}dx^+(dr_1dr_2dr_3dr_4 + dy_1dy_2dy_3dy_4)\,. \tag{3.38}$$

For simplicity let us set the axion to zero $\tau_{1\,s}=0$. When we quantize strings in lightcone gauge with $x^+ = \tau$ we find a massive theory on the string worldsheet. The oscillators in the r directions have the same spectrum as in the standard AdS case, [44]. The oscillators in the y directions have the spectrum

$$\omega_{n\,y\pm} = \sqrt{1 + \left(\frac{n}{|p_-|} \pm \tilde\gamma\right)^2 + \tilde\sigma^2} = \sqrt{1 + (4\pi g_s N)\left|\frac{n}{J} \pm \beta\right|^2} \tag{3.39}$$

where the $\pm$ indicates the spin on the y^1, y^2 or y^2, y^4 planes. In fact, this pp-wave and its relation to the BMN limit of the field theory were studied by Niarchos and Prezas [17]. They started from the field theory and they derived the result (3.39) by generalizing the

[18] Note that it makes sense to compare the normalization of the two point functions because these are marginal operators and the parameter γ has a natural normalization, where it has period $\gamma \sim \gamma + 1$.

arguments in [46]. Then they realized that these results could be reproduced by the pp-wave in (3.35). Here we have shown how (3.35) arises from the full ten dimensional geometry. In general, on the pp-wave we see what we have BPS states whenever $\sigma = 0$ and $n = |p_-|\tilde{\gamma}$. In terms of the original variables we could write $\beta = \gamma = 1/n'$, $J = n'k$ and $n = k$ where k is not necessarily large in the limit (3.34).

We could also take a different pp-wave limit by looking at states having charges near to $(J_1, J_2, J_3) \sim (J, J, J)$. In this case we obtain a different pp-wave. This pp-wave can be obtained by taking the appropriate limit of our metric or by performing an $\mathrm{SL}(2, \mathbb{R})$ transformation on the maximally supersymmetric pp-wave in "magnetic" coordinates. Namely, we start with [47]

$$ds^2 = -dx^+ dx^- - r^2 (dx^+)^2 + d\vec{r}^{\,2} + 4dx^+(y_1 dy_2 + y_3 dy_4) + d\vec{y}^{\,2} \tag{3.40}$$

and then we apply the $\mathrm{SL}(2, \mathbb{R})$ transformation (2.3) by considering a two torus parametrized by y^2, y^4. These coordinates are non-compact, but we can compactify them, perform (2.3), and then decompactify them again. The resulting metric is the pp-wave limit of the solution (3.14) along the null geodesic $\psi = t$. We have not checked whether results computed using this new pp-wave limit agree with expectations from the gauge theory side.

It is natural to ask if the β-deformed field theory leads to an integrable spin chain. In [48, 49] this question was answered in the affirmative in the sector made with chiral fields Φ_1, Φ_2 for generic β or in the sector with all three chiral fields Φ_1, Φ_2, Φ_3 with β real, while is it not integrable for complex β. These results are important for the study of the pp-wave resulting from an $\mathrm{SL}(2, \mathbb{R})$ transformation of (3.40).

4. More general theories

In this section we extend the previous discussion to more general theories. The simplest examples are orbifolds of $AdS_5 \times S^5$. It is clear from our construction of the marginal deformation as a star product that the resulting theories are conformal to leading order in N, since planar diagrams are the same as in the original theory [27]. In the case that we have a Z_n orbifold that preserves at least $\mathcal{N} = 1$ supersymmetry, then the results of [12, 50] show that the marginal deformations of the type we are considering are exactly marginal and preserve $\mathcal{N} = 1$ superconformal invariance. In these cases the Z_n orbifold action is a subgroup of the $\mathrm{U}(1) \times \mathrm{U}(1)$ non-R-symmetry that was used to construct the marginal deformation (1.1). The metrics for these deformations are just simply orbifolds of (3.24).

A conformal field theory that is particularly interesting is the theory living on D3 branes on a conifold [18], see [19] for a review. This theory has an $\mathrm{SU}(2) \times \mathrm{SU}(2)$ global symmetry that leaves the supercharges invariant. We can consider a $\mathrm{U}(1) \times \mathrm{U}(1)$ subgroup of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ and we can apply the previous construction based on this subgroup. In other words, we can deform our theory via (1.1), which leads to the following change in the superpotential[19]

$$\mathrm{Tr}[A_+B_+A_-B_- - A_-B_+A_+B_-] \to \mathrm{Tr}[e^{i\pi\beta}A_+B_+A_-B_- - e^{-i\pi\beta}A_-B_+A_+B_-] \tag{4.1}$$

[19]There is a minor subtlety in this case. In order to get the normalization of β in (4.1), so that its period is $\beta \sim \beta + 1$, we need to define the two U(1) to be $n_{1,2} = J^3_A \pm J^3_B$, where $J^3_{A,B}$ are the SU(2) generators

where $A_\pm$ is in the $(\mathbf{N}, \bar{\mathbf{N}})$ representation and $B_\pm$ is in the $(\bar{\mathbf{N}}, \mathbf{N})$ representation of the gauge group $\mathrm{SU}(N) \times \mathrm{SU}(N)$. This deformation is exactly marginal [20]. The reason is that we have two superpotential parameters, plus the gauge coupling. On the other hand all beta function equations are proportional to $\gamma_{A_-} + \gamma_{A_+} + \gamma_{B_-} + \gamma_{B_+}$, so there is only one constraint. Therefore we have a two dimensional space of solutions.[20] The general analysis of exactly marginal deformations of this field theory is performed in [20]. The fact that there is a corresponding massless field in the gravity solution at lowest order (in β) follows from the analysis of the Kaluza-Klein spectrum in [52, 53]. Another type of marginal deformation, which preserves a single diagonal SU(2) symmetry was considered in [51].

Let us now turn to the gravity side, these two U(1) symmetries act geometrically and we can find the deformed solution by applying the $SL(2,\mathbb{R})$ transformation (2.3) to the $AdS_5 \times T^{1,1}$ solution in [18]. This gives[21]

$$\begin{aligned} ds^2_E &= R^2 G^{-1/4}\Bigg\{ ds^2_{AdS_5} + \frac{s_1^2 s_2^2}{324 f} d\psi^2 + \frac{1}{6}(d\theta_1^2 + d\theta_2^2) + \qquad (4.2) \\ &\qquad + G\left[h\left(d\phi_1 + \frac{c_1 c_2 d\phi_2}{9h} + \frac{c_1 d\psi}{9h}\right)^2 + \frac{f}{h}\left(d\phi_2 + \frac{c_2 s_1^2 d\psi}{54 f}\right)^2\right]\Bigg\} \\ e^{2\phi} &= e^{2\phi_0} G \\ B^{NS} &= 2\gamma R^4 G f \left(d\phi_1 + \frac{c_1 c_2 d\phi_2}{9h} + \frac{c_1 d\psi}{9h}\right) \wedge \left(d\phi_2 + \frac{c_2 s_1^2 d\psi}{54 f}\right) \\ B^{RR} &= \gamma \frac{\pi N}{2} c_1 s_2 d\theta_2 d\psi\,, \\ F^{(5)} &= 27\pi N(\omega_{AdS} + *\omega_{AdS}) \end{aligned}$$

$$G^{-1} \equiv 1 + \gamma^2 4 R^4 f\,, \qquad R^4 = \frac{27}{4}\pi e^{\phi_0} N \qquad (4.3)$$

$$h \equiv \frac{c_1^2}{9} + \frac{s_1^2}{6}\,, \qquad f \equiv \frac{1}{54}(c_2^2 s_1^2 + c_1^2 s_2^2) + \frac{s_1^2 s_2^2}{36} \qquad (4.4)$$

We have presented here the solution when β is real ($\beta = \gamma$). The general solution can be found in a similar way, see appendix A. In fact, most of the discussion that we had in the case of $AdS_5 \times S^5$ goes through here with minor modifications. Namely, the discussion on the coulomb branch, as well as some of the discussion on the BPS chiral primaries. For example, we can also compute the Zamolodchikov metric for the marginal deformations to get

$$S = \frac{27 N^2}{2^8 \pi^2} \int \left[\frac{\partial\tau_s \partial\bar{\tau}_s}{\tau_{2\,s}^2} + C\frac{|\partial\gamma - \tau_s \partial\sigma|^2}{\tau_{2\,s}} \right] \qquad (4.5)$$

---

that act on the fields A, B and they have a half integer quantization condition. The two new combinations we defined have integer quantization when they act on gauge invariant operators. These new generators are the ones that appear in our general formulas (1.1), etc.

[20] We thank I. Klebanov for this argument.

[21] We denote $c_1 \equiv \cos\theta_1$, $s_2 = \sin\theta_2$, etc. In order to understand some of the numerical factors in the following formula it is necessary to remember that we are applying our general procedure on the two torus parametrized by $\varphi^{1,2} = \frac{\phi^1 \pm \phi^2}{2}$, as we explained in a previous footnote.

where as in (3.33)

$$C = \langle g_{0,E} \rangle = 4R_E^4 \langle f \rangle = 4R_E^4 \left\langle \frac{1}{54}(c_2^2 s_1^2 + c_1^2 s_2^2) + \frac{s_1^2 s_2^2}{36} \right\rangle = 4R_E^4 \frac{5}{243} = \frac{7\pi}{16} N \,. \quad (4.6)$$

Notice that CFT's which are obtained from D-branes at singularities of toric Calabi-Yau's have three U(1) symmetries. Two of them leave the supercharge invariant. The gravity solutions corresponding to these theories [21, 22] have the form of $AdS_5 \times Y_{p,q}$, where the $Y_{p,q}$ spaces were found in [54]. The two U(1) symmetries are isometries of $Y_{p,q}$. So we can apply our $\mathrm{SL}(2,\mathbb{R})$ method to deform both the field theory solution and the gravity solution. These deformations are exactly marginal [20]. The resulting solutions are presented in appendix A.

Note that our method based on the transformation (2.3) is not limited to conformal field theories. In fact one can also apply it for non-conformal field theories. The new feature that arises is that the region of validity of the gravity solution might depend on the radial coordinate, as in [31]. The reason is that through the transformation (2.3) a problem arises if the original volume of the two torus becomes too large in string units. In that case we see that the transformed volume (2.3) becomes very small so that the gravity solution might become invalid. This can happen even in a region where the original solution is non singular. In fact, a simple example is the metric (2.5) which becomes problematic at large distances. In fact, for the conformal case this was basically the origin of the first condition in (3.16). For example, we can consider the gravity solution corresponding to a D-p-brane field theory for $p < 3$ and then apply our method. More specifically, let us consider the field theory on D2 branes and perform this trick with two U(1)s in $\mathrm{SU}(3) \subset \mathrm{SO}(7)$. We get a supersymmetric theory which is the dimensional reduction of the β-deformed theory to 2+1 dimensions. The deformation is no longer marginal, in fact in the UV, which corresponds to weakly coupled Yang Mills, it is a relevant deformation of dimension 5/2. The gravity solution will involves factors of

$$G^{-1} = 1 + \gamma^2 f r^4 \sim 1 + \gamma^2 \frac{1}{r} \,, \qquad f \sim \frac{1}{r^5} \quad (4.7)$$

where f is the harmonic function appearing in the D2 gravity dual [31]. We see that also for strong 't Hooft coupling the deformation becomes important in the IR. Note that in the supergravity regime this theory is not conformal, nevertheless we can say that the effects of the deformation become stronger at shorter distance scales, relative to the effects of the gauge coupling. It would be nice to see if there is some valid supergravity description for the IR theory.

Another interesting example is the cascading theory studied by Klebanov and Strassler [23]. We can deform both the field theory and the gravity solution by applying (2.3), (1.1). The final gravity solution looks quite messy and is written in appendix A. Here let us summarize just a couple of features. Suppose that we make a deformation with a small but fixed γ. Then the gravity solution in the UV region becomes strongly curved if we go far enough. The reason is that in the UV region of the original solution [23] the radius of (the approximately) $T^{1,1}$ is growing, so that the transformation (2.3) might generate a highly curved space. Another feature of the solution concerns the IR region. In the IR region the undeformed solution [23] has a finite size S^3. Our deformation does not vanish in the IR region of the geometry, it deforms the three sphere. This might be useful for getting theories that are closer to pure $\mathcal{N}=1$ super-Yang-Mills.

In appendix A we give the general action of the SL$(2,\mathbb{R})$ transformation, so that it becomes a simple matter to generate new solutions.

Notice that there are some cases where our trick cannot be applied because some of the ingredients are not present. For example, if we consider an $\mathcal{N}=4$ field theory with SO(N) gauge group, then there are no β-deformations since the operator in question vanishes identically due to the anti-symmetry of the matrices in the adjoint representation. Correspondingly, in the gravity solution there is a two torus, but the B field on this two torus is projected out so that we cannot form the τ parameter discussed above.

5. Deformations based on U$(1)^3$ symmetries

Suppose that we start with M-theory compactified on a three torus. Then, as we saw above we have an SL$(3,\mathbb{R})\times$SL$(2,\mathbb{R})$ symmetry. So far we have been using a transformation (2.3) in the SL$(3,\mathbb{R})$ subgroup. A natural question to ask is whether something interesting can be done with the SL$(2,\mathbb{R})$ subgroup. In other words, we are interested in applying (2.3) to the parameter

$$\tau = C_{123} + i\sqrt{G} \tag{5.1}$$

where C_{123} is the value of the C field over the two torus and $\sqrt{G}$ is the volume of the three torus.

Indeed one can generate new solutions in this fashion. For example, we can start with the gravity solution describing coincident M2 branes. We can choose three U(1) symmetries that corresponding to

$$\begin{aligned}(z_1,z_2,z_3,z_4) &\to (z_1,z_2,e^{-i\varphi_1}z_3,e^{i\varphi_1}z_4)\\ &\to (z_1,e^{-i\varphi_2}z_2,e^{i\varphi_2}z_3,z_4)\\ &\to (e^{i\varphi_3}z_1,e^{-i\varphi_3}z_2,z_3,z_4)\,.\end{aligned} \tag{5.2}$$

These symmetries are realized geometrically on $AdS_4\times S^7$. We see that the deformation based on (2.3) with (5.1) gives a new gravity solution in which the S^7 is smoothly deformed. Since the transformations (5.2) are embedded in SU(4) $\subset$ SO(8) so they will preserve two supersymmetries in three dimensions.

On the field theory side this deformation corresponds to turning on an operator in the fourth spherical harmonic of SO(8) (symmetric traceless representation) which is also in the fourth fold symmetric representation of SU(4) and is U$(1)^3$ invariant. There is only one such state, which we can think of as a spherical harmonic of the form $z_1z_2z_3z_4$. This should be interpreted simply as giving the quantum numbers of the operator in the field theory. Other marginal deformations were considered in [58]. The metric is

$$ds_{11}^2 = G^{-1/3}R^2\left[\frac{1}{4}ds^2_{AdS} + \sum_{i=1}^4(d\mu_i^2 + G\mu_i^2 d\phi_i^2) + 16G\hat{\gamma}^2\mu_1^2\mu_2^2\mu_3^2\mu_4^2\Big(\sum d\phi_i\Big)^2\right]$$

$$
\begin{aligned}
F_{(4)} &= \frac{3}{8}R^3(\omega_{AdS} + 16\hat{\gamma}s_\theta^5 c_\theta s_{2\alpha}^2 s_{2\beta} d\theta d\alpha d\beta d\psi) + \qquad (5.3)\\
&\quad + R^3\hat{\gamma} d\left\{ G\Big(\prod \mu_i^2\Big) \left[\frac{d\phi_1 d\phi_2 d\phi_3}{\mu_4^2} + \frac{d\phi_1 d\phi_3 d\phi_4}{\mu_2^2} - \frac{d\phi_1 d\phi_2 d\phi_4}{\mu_3^2} - \frac{d\phi_2 d\phi_3 d\phi_4}{\mu_1^2} \right] \right\}\\
\Delta &= \mu_1^2\mu_2^2\mu_3^2\mu_4^2 \sum_1^4 \mu_i^{-2}\,, \qquad G^{-1} = 1 + \hat{\gamma}^2\Delta\,, \quad R = (32\pi^2 N)^{1/6}\,, \qquad \hat{\gamma} \equiv \gamma R^3\,. \quad (5.4)
\end{aligned}
$$

6. Conclusions

In this paper we studied a particular deformation of a field theory. This deformation amounts to a redefinition of the product of fields in the lagrangian by introducing phases which depend on the order of fields that are charged under a global $U(1) \times U(1)$ symmetry. Then we can easily do two things: First, if we know that D-branes on some background give us the undeformed field theory, then we can deform the background by performing the $SL(2, \mathbb{R})$ transformation (2.3) in order to get a new background such that D-branes on this new background give us the deformed field theory. Second, if we know the gravity dual of the field theory, and the $U(1) \times U(1)$ symmetry is realized geometrically, then we can perform the transformation (2.3) to get the new background geometry.

This procedure works both for conformal and non-conformal cases. If the original theory is supersymmetric, then the deformed theory will be supersymmetric if the $U(1) \times U(1)$ symmetry commutes with the supercharge (i.e. they are not R-symmetries).

It seems that more general dualities might enable us to find very easily other interesting gravity solutions.

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A. Solution generating technique

In this appendix we summarize the technique which was used to generate supergravity solutions presented in this paper.

We begin with some solution of 11 dimensional supergravity which has a T^3 symmetry. This KK reduction was considered in [55, 56][22]

$$
ds^2 = \Delta^{1/3} M_{ab} \mathcal{D}\varphi^a \mathcal{D}\varphi^b + \Delta^{-1/6} g_{\mu\nu} dx^\mu dx^\nu
$$

[22] In [55] they reduced on S^3 and found a gauged supergravity. We can take the limit when the gauge coupling goes to zero to get the reduction on a torus. See [56] for more general reductions.

$$
\begin{aligned}
C^{(3)} &= \frac{1}{2}(C_{a\mu\nu}\mathcal{D}\varphi^a \wedge dx^\mu \wedge dx^\nu + C_{ab\nu}\mathcal{D}\varphi^a \wedge \mathcal{D}\varphi^b \wedge dx^\mu) + \\
&\quad + \frac{1}{6}C_{\mu\nu\lambda}dx^\mu \wedge dx^\nu \wedge dx^\lambda + C\mathcal{D}\varphi^1\mathcal{D}\varphi^2\mathcal{D}\varphi^3
\end{aligned} \tag{A.1}
$$

where the indices $a, b = 1, 2, 3$ and the indices $\mu, \nu, \ldots$ run over the remaining eight dimensions. The determinant of the matrix M is one. Here we use notation $\mathcal{D}\varphi^a = d\varphi^a + \mathcal{A}^a_\mu dx^\mu$ for $a = 1, 2, 3$. Under a general $\mathrm{SL}(3, \mathbb{R})$ transformation of coordinates $(\varphi^1, \varphi^2, \varphi^3)$:

$$
\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}' = (\Lambda^T)^{-1} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} \tag{A.2}
$$

the fields transform in a following way

$$
\begin{aligned}
&M \to \Lambda M \Lambda^T, \qquad \begin{pmatrix} C_{23\mu} \\ C_{31\mu} \\ C_{12\mu} \end{pmatrix} \to (\Lambda^T)^{-1} \begin{pmatrix} C_{23\mu} \\ C_{31\mu} \\ C_{12\mu} \end{pmatrix}, \\
&\begin{pmatrix} \mathcal{A}^1_\mu \\ \mathcal{A}^2_\mu \\ \mathcal{A}^3_\mu \end{pmatrix} \to (\Lambda^T)^{-1} \begin{pmatrix} \mathcal{A}^1_\mu \\ \mathcal{A}^2_\mu \\ \mathcal{A}^3_\mu \end{pmatrix}, \qquad \begin{pmatrix} C_{1\mu\nu} \\ C_{2\mu\nu} \\ C_{3\mu\nu} \end{pmatrix} \to \Lambda \begin{pmatrix} C_{1\mu\nu} \\ C_{2\mu\nu} \\ C_{3\mu\nu} \end{pmatrix}.
\end{aligned} \tag{A.3}
$$

In this framework the $\mathrm{SL}(3, \mathbb{R})$ symmetry is manifest. Now we perform a reduction along φ^3 to produce a solution of type-IIA supergravity. At this step it is convenient to write the matrix M in (A.1) as

$$
M_{ab}\mathcal{D}\varphi^a\mathcal{D}\varphi^b = e^{-2\phi/3}h_{mn}\mathcal{D}\varphi^m\mathcal{D}\varphi^n + e^{4\phi/3}(\mathcal{D}\varphi^3 + N_m\mathcal{D}\varphi^m)^2 \tag{A.4}
$$

where $m, n = 1, 2$ and the determinant of h_{ab} is one. After we find the IIA solution, we perform a T duality along φ^1 to produce a solution of type-IIB supergravity

$$
\begin{aligned}
ds^2_{IIB} &= \frac{1}{h_{11}}\left[\frac{1}{\sqrt{\Delta}}(D\varphi_1 - CD\varphi^2)^2 + \sqrt{\Delta}(D\varphi^2)^2\right] + e^{2\phi/3}g_{\mu\nu}dx^\mu dx^\nu, \\
B &= \frac{h_{12}}{h_{11}}D\varphi^1 \wedge D\varphi^2 - C_{32\mu}D\varphi^2 \wedge dx^\mu + D\varphi^1 \wedge \mathcal{A}^1 - \frac{1}{2}C_{3\mu\nu}dx^\mu \wedge dx^\nu + C_{31\mu}dx^\mu \wedge \mathcal{A}^1 \\
e^{2\Phi} &= \frac{e^{2\phi}}{h_{11}}, \qquad C^{(0)} = N_1 \\
C^{(2)} &= -(N_2 - \frac{h_{12}}{h_{11}}N_1)D\varphi^1 \wedge D\varphi^2 - C_{12\mu}D\varphi^2 \wedge dx^\mu - D\varphi^1 \wedge \mathcal{A}^3_\mu dx^\mu - \\
&\quad -\frac{1}{2}C_{1\mu\nu}dx^\mu \wedge dx^\nu + C_{31\mu}dx^\mu \wedge \mathcal{A}^3_\nu \\
C^{(4)} &= -\left(\frac{1}{2}[C_{2\mu\nu} + 2C_{32\mu}\mathcal{A}^3_\nu - \frac{h_{12}}{h_{11}}(C_{1\mu\nu} + 2C_{31\mu}\mathcal{A}^3_\nu)]D\varphi^2 dx^\mu dx^\nu + \right. \\
&\qquad \left. + \frac{1}{6}(C_{\mu\nu\lambda} + 3C_{3\mu\nu}\mathcal{A}^3_\lambda)dx^\mu dx^\nu dx^\lambda\right) \wedge D\varphi^1 + \\
&\quad + d_{\mu_1\mu_2\mu_3\mu_4}dx^{\mu_1}dx^{\mu_2}dx^{\mu_3}dx^{\mu_4} + \hat{d}_{\mu_1\mu_2\mu_3}dx^{\mu_1}dx^{\mu_2}dx^{\mu_3}D\varphi^2 \\
D\varphi^1 &= d\varphi^1 - C_{31\mu}dx^\mu, \qquad D\varphi_2 = d\varphi^2 + \mathcal{A}^2_\mu dx^\mu.
\end{aligned} \tag{A.5}
$$

The forms $d_{\mu_1\mu_2\mu_3\mu_4}$, $\hat{d}_{\mu_1\mu_2\mu_3}$ are determined by the self duality conditions for the five form field strength. In order to find them, we need to impose the equations of motion for $C_{\mu\nu\lambda}$ and $C_{2\mu\nu}$. These variables were natural from the eleven dimensional point of view, but they are not so natural from the point of view of IIB theory. Let us rewrite the solution as

$$\begin{aligned}
ds^2_{IIB} &= F\left[\frac{1}{\sqrt{\Delta}}(D\varphi_1 - C(D\varphi^2))^2 + \sqrt{\Delta}(D\varphi_2)^2\right] + \frac{e^{2\phi/3}}{F^{1/3}} g_{\mu\nu} dx^\mu dx^\nu \,, \\
B &= B_{12}(D\varphi^1)\wedge(D\varphi^2) + \left\{B_{1\mu}(D\varphi^1) + B_{2\mu}(D\varphi^2)\right\}\wedge dx^\mu - \\
&\quad -\frac{1}{2}A^m_\mu B_{m\nu} dx^\mu \wedge dx^\nu + \frac{1}{2}\tilde{b}_{\mu\nu} dx^\mu \wedge dx^\nu \\
e^{2\Phi} &= e^{2\phi}\,, \qquad C^{(0)} = \chi \\
C^{(2)} &= C_{12}(D\varphi^1)\wedge(D\varphi^2) + \left\{C_{1\mu}(D\varphi^1) + C_{2\mu}(D\varphi^2)\right\}\wedge dx^\mu \\
&\quad -\frac{1}{2}A^m_\mu C_{m\nu} dx^\mu \wedge dx^\nu + \frac{1}{2}\tilde{c}_{\mu\nu} dx^\mu \wedge dx^\nu \\
C^{(4)} &= -\frac{1}{2}(\tilde{d}_{\mu\nu} + B_{12}\tilde{c}_{\mu\nu} - \epsilon^{mn} B_{m\mu} C_{n\nu} - B_{12} A^m_\mu C_{m\nu}) dx^\mu dx^\nu D\varphi^1 D\varphi^2 + \\
&\quad +\frac{1}{6}(C_{\mu\nu\lambda} + 3(\tilde{b}_{\mu\nu} + A^1_\mu B_{1\nu} - A^2_\mu B_{2\nu}) C_{1\lambda}) dx^\mu dx^\nu dx^\lambda D\varphi^1 + \qquad \text{(A.6)} \\
&\quad +d_{\mu_1\mu_2\mu_3\mu_4} dx^{\mu_1} dx^{\mu_2} dx^{\mu_3} dx^{\mu_4} + \hat{d}_{\mu_1\mu_2\mu_3} dx^{\mu_1} dx^{\mu_2} dx^{\mu_3} D\varphi^2 \qquad \text{(A.7)}
\end{aligned}$$

where

$$D\varphi^2 = d\varphi^2 + A^2\,, \qquad D\varphi^1 = d\varphi^1 + A^1\,. \tag{A.8}$$

When we wrote (A.6) we relabelled various fields and we have shifted the two forms in eight dimensions by an SL$(3,\mathbb{R})$ covariant combination of the one forms in eight dimensions. Then we can deduce the action of SL$(3,\mathbb{R})$ transformations. We have three objects which transform as vectors:

$$V^{(1)}_\mu = \begin{pmatrix} -B_{2\mu} \\ A^1_\mu \\ C_{2\mu} \end{pmatrix}, \quad V^{(2)}_\mu = \begin{pmatrix} B_{1\mu} \\ A^2_\mu \\ -C_{1\mu} \end{pmatrix} : \quad V^{(i)}_\mu \to (\Lambda^T)^{-1} V^{(i)}_\mu\,; \tag{A.9}$$

$$W_{\mu\nu} = \begin{pmatrix} \tilde{c}_{\mu\nu} \\ \tilde{d}_{\mu\nu} \\ \tilde{b}_{\mu\nu} \end{pmatrix} \to \Lambda W_{\mu\nu} \tag{A.10}$$

and one matrix

$$M = gg^T\,, \quad g^T = \begin{pmatrix} e^{-\phi/3}F^{-1/3} & 0 & 0 \\ 0 & e^{-\phi/3}F^{2/3} & 0 \\ 0 & 0 & e^{2\phi/3}F^{-1/3} \end{pmatrix} \begin{pmatrix} 1 & B_{12} & 0 \\ 0 & 1 & 0 \\ \chi & -C_{12} + \chi B_{12} & 1 \end{pmatrix} \tag{A.11}$$

which transforms as

$$M \to \Lambda M \Lambda^T\,. \tag{A.12}$$

The scalars Δ, C as well as the three form $C_{\mu\nu\lambda}$ stay invariant under these $\mathrm{SL}(3,\mathbb{R})$ transformations. Under $\mathrm{SL}(2,\mathbb{R})$ transformations $C + i\sqrt{\Delta}$ transform as a τ parameter. Then the one forms (A.9) transform into each other, the two forms (A.10) remain invariant and the fourform field strength in eight dimensions coming from $C_{\mu\nu\delta}$ in the last line of (A.7) transforms into its magnetic dual in eight dimensions. The metric $g_{\mu\nu}$ is the Einstein metric in eight dimensions and does not change under any of the transformations.

Let us consider a particular example of the $\mathrm{SL}(3,\mathbb{R})$ transformation. We begin with geometry which has only metric and $\tilde{d}_{\mu\nu}$ excited, with all other fields set to zero, including the dilaton. This implies that the five-form field strength can be written as

$$F_{(5)} = \tilde{F}_{(5)} + {}^*\tilde{F}_{(5)} \tag{A.13}$$

where $\tilde{F}_{(5)}$ has no indices along the torus (φ^1, φ^2). Let us apply the transformation with[23]

$$\Lambda = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & \sigma & 1 \end{pmatrix}. \tag{A.14}$$

Then we find the transformed fields

$$g'^T = \begin{pmatrix} \frac{F^{-1/3}}{\sqrt{GH}} & 0 & 0 \\ 0 & F^{2/3}\sqrt{G} & 0 \\ 0 & 0 & F^{-1/3}\sqrt{H} \end{pmatrix} \begin{pmatrix} 1 & \gamma F^2 G & 0 \\ 0 & 1 & 0 \\ \gamma\sigma\frac{F^2}{H} & \sigma\frac{F^2}{H} & 1 \end{pmatrix}.$$

By comparing this with (A.11) we can find the new fields

$$\begin{gathered} B'_{12} = \gamma F^2 G\,, \quad C'_{12} = -\sigma F^2 G\,, \quad \chi' = \gamma\sigma\frac{F^2}{H}\,, \quad F' = FG\sqrt{H}\,, \quad e^{2\phi'} = GH^2 \\ G^{-1} \equiv 1 + (\gamma^2 + \sigma^2)F^2\,, \quad H = 1 + \sigma^2 F^2 \end{gathered} \tag{A.15}$$

and the new geometry

$$\begin{aligned} ds^2_{IIB} &= FGH^{1/2}\left[\frac{1}{\sqrt{\Delta}}(D\varphi_1 - C(D\varphi^2))^2 + \sqrt{\Delta}(D\varphi_2)^2\right] + H^{1/2}F^{-1/3}g_{\mu\nu}dx^\mu dx^\nu\,, \\ B &= \gamma F^2 G(D\varphi^1)\wedge(D\varphi^2) + \frac{\sigma}{2}\tilde{d}_{\mu\nu}dx^\mu\wedge dx^\nu\,, \qquad e^{2\phi} = H^2 G \\ C^{(2)} &= -\sigma F^2 G(D\varphi^1)\wedge(D\varphi^2) + \frac{\gamma}{2}\tilde{d}_{\mu\nu}dx^\mu\wedge dx^\nu\,, \qquad \chi = \gamma\sigma F^2 H^{-1}\,, \\ F^{(5)} &= \tilde{F}_{(5)} + {}^*\tilde{F}_{(5)}\,. \end{aligned} \tag{A.16}$$

Notice that $\tilde{F}_{(5)}$ is the same as before, but the star is now taken with the new metric.

[23] The matrix h_3 used in (2.10) (with $h_2 = 1$) is equivalent to (A.14) after interchanging the first two rows and columns.

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As an example we consider an application this of procedure to flat space. We begin with metric on R^{10} which we write in a form

$$
\begin{aligned}
ds^2 &= \eta_{\mu\nu}dx^\mu dx^\nu + \sum_{i=1}^{3} dr_i^2 + r_1^2(d\psi - d\varphi_2)^2 + r_2^2(d\psi + d\varphi_1 + d\varphi_2)^2 + r_3^2(d\psi - d\varphi_1)^2 \\
&= \eta_{\mu\nu}dx^\mu dx^\nu + \sum dr_i^2 + (r_2^2 + r_3^2)\left(D\varphi_1 + \frac{r_2^2}{r_2^2 + r_3^2}D\varphi_2\right)^2 + \\
&\quad + \frac{g_0}{r_2^2 + r_3^2}(D\varphi_2)^2 + \frac{9r_1^2 r_2^2 r_3^2}{g_0}d\psi^2
\end{aligned}
$$

where we defined

$$
\begin{aligned}
g_0 &\equiv r_1^2 r_2^2 + r_1^2 r_3^2 + r_2^2 r_3^2 \,, \\
D\varphi_1 &= d\varphi_1 - d\psi + \frac{3r_1^2 r_2^2}{g_0}d\psi \,, \qquad D\varphi_2 = d\varphi_1 - d\psi + \frac{3r_2^2 r_3^2}{g_0}d\psi \,. \qquad \text{(A.17)}
\end{aligned}
$$

Then transformation with parameter γ gives a new solution of type-IIB supergravity

$$
\begin{aligned}
ds^2_{IIB} &= \eta_{\mu\nu}dx^\mu dx^\nu + \sum dr_i^2 + \\
&\quad + G[r_1^2(d\psi - d\varphi_1)^2 + r_2^2(d\psi + d\varphi_1 + d\varphi_2)^2 + r_3^2(d\psi - d\varphi_2)^2 + 9\gamma^2 g_0 r_1^2 r_2^2 r_3^2 d\psi^2] \\
B &= \gamma g_0 G(D\varphi^1) \wedge (D\varphi^2) \quad e^{2\Phi} = G \,, \qquad G^{-1} \equiv 1 + \gamma^2 g_0 \,.
\end{aligned}
$$

This background preserves two supersymmetries in four dimensions, namely the two that transform as a singlet of SU(3) acting on the three complex coordinates of R^6. If we place D3 branes at the origin of this space, $r_i = 0$, and take a low energy limit we find the β-deformation of $\mathcal{N} = 4$ super Yang Mills. Application of this procedure to $AdS_5 \times S^5$ gives the solution (3.24).

Let us discus now the regularity of the transformed solution. Suppose that we start with a metric that is non-singular as a ten dimensional theory. When do we get a non-singular solution?. In principle there are a variety of things that could go wrong. For example, the original theory could be such that τ, in (2.2) is only defined up to gauge transformations of the B field. Then the final geometry will not be well defined. So τ has to be globally well defined and should be such that $\tau_1 \to 0$ when $\tau_2 \to 0$. In the case that the original B field is non-zero we also should worry about the components of the B field that are vectors in eight dimensions. In eight dimensions these vectors can have Wilson lines, which if integer, are allowed by the regularity conditions. However, under the transformation the field $A^1_\mu \to A^1_\mu + \gamma B_{2\mu}$ and this can lead to non-integer Wilson lines. So this is another thing that needs to be checked. When we check these properties we are allowed to redefine the coordinates φ^m in such a way to make our job easier. This amounts to gauge transformations of the A^m_μ fields. It turns out that performing this transformations before or after the $SL(2,\mathbb{R})$ transformation does not change the final answer. So the conclusion is that A need not be globally defined and it need not have vanishing Wilson lines, as long as the original metric is regular and the B_μ and C_μ gauge fields are globally well defined one forms. These remarks are useful for checking the regularity of the deformed

Klebanov-Strassler solution. Another potential problem is the fact that the last terms in the B and $C^{(2)}$ fields in (A.16) might lead to fluxes that are not properly quantized. An example were we would run into a problem arises if we have a $T^2 \times S^3$ with N units of flux of F_5 in the original geometry. Then, for general γ, we would have a non-quantized $dC^{(2)}$ flux on the S^3, which is not allowed. We have checked on a case by case basis that this does not happen for our solutions.

A.1 Gravity dual of the conifold conformal field theory.

It is straightforward to apply the procedure outlined above to $AdS_5 \times T^{1,1}$.

$$\begin{aligned} \frac{ds_E{}^2}{R_E^2} &= ds^2_{AdS} + \frac{1}{9}(d\psi + c_1 d\phi_1 + c_2 d\phi_2)^2 + \frac{1}{6}(s_1^2 d\phi_1^2 + s_2^2 d\phi_2^2 + d\theta_1^2 + d\theta_2^2) \\ F^{(5)} &= 4R_E^4(\omega_{AdS} +^* \omega_{AdS}) \,. \end{aligned} \tag{A.18}$$

We rewrite it to make the T^2 part more explicit

$$\begin{aligned} \frac{ds^2}{R^2} &= h\left(d\phi_1 + \frac{c_1 c_2 d\phi_2}{9h} + \frac{c_1 d\psi}{9h}\right)^2 + \frac{f}{h}\left(d\phi_2 + \frac{c_2 s_1^2 d\psi}{54 f}\right)^2 + \frac{s_1^2 s_2^2}{324 f} d\psi^2 + \\ &\quad + \frac{1}{6}(d\theta_1^2 + d\theta_2^2) + ds^2_{AdS} \end{aligned} \tag{A.19}$$

$$h \equiv \frac{c_1^2}{9} + \frac{s_1^2}{6}\,, \qquad f \equiv \frac{1}{54}(c_2^2 s_1^2 + c_1^2 s_2^2) + \frac{s_1^2 s_2^2}{36}\,. \tag{A.20}$$

This implies that the deformed metric is simply

$$\begin{aligned} \frac{ds_E^2}{R_E^2} &= G^{3/4}\left[h\left(d\phi_1 + \frac{c_1 c_2 d\phi_2}{9h} + \frac{c_1 d\psi}{9h}\right)^2 + \frac{f}{h}\left(d\phi_2 + \frac{c_2 s_1^2 d\psi}{54 f}\right)^2\right] + \qquad \text{(A.21)} \\ &\quad + G^{-1/4}\left[\frac{s_1^2 s_2^2}{324 f} d\psi^2 + \frac{1}{6}(d\theta_1^2 + d\theta_2^2) + ds^2_{AdS}\right] \\ \frac{B}{R_E^2} &= \hat\gamma G f \left(d\phi_1 + \frac{c_1 c_2 d\phi_2}{9h} + \frac{c_1 d\psi}{9h}\right) \wedge \left(d\phi_2 + \frac{c_2 s_1^2}{54 f} d\psi\right) + \frac{\hat\sigma}{27} c_1 s_2 d\theta_2 d\psi\,, \\ \frac{C^{(2)}}{R_E^2} &= -\hat\sigma G f \left(d\phi_1 + \frac{c_1 c_2 d\phi_2}{9h} + \frac{c_1 d\psi}{9h}\right) \wedge \left(d\phi_2 + \frac{c_2 s_1^2}{54 f} d\psi\right) + \frac{\hat\gamma}{27} c_1 s_2 d\theta_2 d\psi\,, \\ \frac{F^{(5)}}{R_E^4} &= 4(\omega_{AdS} + *\omega_{AdS}) \\ e^{2\Phi} &= G H^2\,, \qquad \chi = \hat\gamma \hat\sigma f H^{-1} \qquad \text{(A.22)} \\ G^{-1} &\equiv 1 + (\hat\sigma^2 + \hat\gamma^2) f\,, \qquad H \equiv 1 + \hat\sigma^2 f\,, \qquad \hat\gamma \equiv 2\gamma R_E^2\,, \qquad \hat\sigma \equiv 2\sigma R_E^2\,. \qquad \text{(A.23)} \end{aligned}$$

Here we presented the solution when $\tau_s = i$. For general τ_s we get a solution similar to (3.24), so that it is hopefully obvious to the reader how to introduce the τ_s dependence.

A.2 Marginal deformations of theories associated to the $\mathrm{Y}^{p,q}$ manifolds

As another example we consider the deformations of recently discovered Sasaki-Einstein spaces $\mathrm{Y}^{p,q}$ [54, 22]

$$ds^2 = ds_{AdS_5} + \frac{1-y}{6}(d\theta^2 + s_\theta^2 d\phi^2) + \frac{1}{w(y)q(y)} dy^2 + \frac{q(y)}{9}(d\psi - c_\theta d\phi)^2$$

$$+w(y)\left[\ell d\alpha+f(y)(d\psi-c_\theta d\phi)\right]^2 \tag{A.24}$$

where

$$w(y)=\frac{2(a-y^2)}{1-y}\,,\qquad q(y)=\frac{a-3y^2+2y^3}{a-y^2}\,,\qquad f(y)=\frac{a-2y+y^2}{6(a-y^2)} \tag{A.25}$$

$$\ell=\frac{q}{3q^2-2p^2+p(4p^2-3q^2)^{1/2}} \tag{A.26}$$

where we are using the same notation is in [22], except that our α has period 2π (and we have set their $c=1$). We identify the two U(1) symmetries as the symmetries shifting α and shifting ϕ. Indeed the holomorphic three form in [22] is invariant under such shifts. Rearranging the metric to make the T^2 more explicit, we get

$$\begin{aligned}ds^2 &= ds_{AdS_5}+\frac{1-y}{6}d\theta^2+\frac{1}{w(y)q(y)}dy^2+\frac{q(y)(1-y)s_\theta^2 d\psi^2}{3(2q(y)c_\theta^2+3s_\theta^2(1-y))}+\\ &\quad +w(y)\left[\ell d\alpha+f(y)(d\psi-c_\theta d\phi)\right]^2+\frac{2q(y)c_\theta^2+3(1-y)s_\theta^2}{18}\times\\ &\quad\times\left(d\phi-\frac{2q(y)c_\theta d\psi}{2q(y)c_\theta^2+3s_\theta^2(1-y)}\right)^2\end{aligned} \tag{A.27}$$

$$F_{(5)}=\omega_{AdS}+\frac{\ell}{18}(1-y)s_\theta\, d\theta dy d\psi d\alpha d\phi\,. \tag{A.28}$$

Then application of our procedure gives the deformed solution[24]

$$\begin{aligned}ds_E^2 &= R_E^2G^{-1/4}\Bigg\{\Bigg[ds_{AdS_5}+\frac{1-y}{6}d\theta^2+\frac{1}{w(y)q(y)}dy^2+\\ &\qquad+\frac{q(y)\ell^2}{27}G(\gamma^2+\sigma^2)s_\theta^2(a-y^2)d\psi^2\Bigg]+\\ &\qquad+G\Bigg[\frac{1-y}{6}s_\theta^2d\phi^2+\frac{q(y)}{9}(d\psi-c_\theta d\phi)^2+\\ &\qquad\qquad+w(y)[\ell d\alpha+f(y)(d\psi-c_\theta d\phi)]^2\Bigg]\Bigg\}\end{aligned} \tag{A.29}$$

$$\begin{aligned}B &= \gamma g_0 R_E^4 G(D\alpha)\wedge(D\phi)+\frac{\pi N}{\mathcal{V}}\frac{8\sigma\ell}{9}(1-y)c_\theta\, dyd\psi\,,\qquad e^{2\phi}=H^2G\\ C^{(2)} &= -\sigma g_0 R_E^4 G(D\alpha)\wedge(D\phi)+\frac{\pi N}{\mathcal{V}}\frac{8\gamma\ell}{9}(1-y)c_\theta\, dyd\psi\,,\qquad \chi=\gamma\sigma g_0H^{-1}\,,\\ F^{(5)} &= \frac{16\pi N}{\mathcal{V}}(\omega_{AdS}+^*\omega_{AdS})\end{aligned} \tag{A.30}$$

$$\begin{aligned}&G^{-1}=1+(\gamma^2+\sigma^2)g_0\,,\qquad H=1+\sigma^2g_0\,,\\ &g_0=\frac{2q(y)c_\theta^2+3(1-y)s_\theta^2}{9(1-y)}(a-y^2)\ell^2\\ &D\alpha=d\alpha+\frac{1}{\ell}\frac{3f(y)s_\theta^2(1-y)d\psi}{2q(y)c_\theta^2+3s_\theta^2(1-y)}\,,\qquad D\phi=d\phi-\frac{2q(y)c_\theta d\psi}{2q(y)c_\theta^2+3s_\theta^2(1-y)}\,.\end{aligned} \tag{A.31}$$

[24]Here we restored the scales associated with the geometry.

Here we defined the ratio of volumes $\mathcal{V}$ (see [54] for details)

$$\mathcal{V} \equiv \frac{\mathrm{vol}(Y^{p,q})}{\mathrm{vol}(S^5)} = \frac{q^2[2p + (4p^2 - 3q^2)^{1/2}]}{3p^2[3q^2 - 2p^2 + p(4p^2 - 3q^2)^{1/2}]}\,, \qquad R_E^4 = \frac{4\pi N}{\mathcal{V}}\,, \tag{A.32}$$

A.3 Deformations of the Klebanov-Strassler solution

Our procedure can be applied to non-conformal theories as well. An example of the gravity solution corresponding to such theory is the Klebanov-Strassler background [23][25]

$$\begin{aligned}
ds^2 &= h^{-1/2} m^2 dx_m dx_m + \\
&\quad + h^{1/2} \frac{3^{1/3}}{2^{4/3}} K \left[\frac{1}{3K^3}(d\tau^2 + (g_5)^2) + \cosh^2\frac{\tau}{2}[(g_3)^2 + (g_4)^2] + \sinh^2\frac{\tau}{2}[(g_1)^2 + (g_2)^2] \right] \\
G_{(3)} &= \frac{M}{2}\left[g_5 \wedge g_3 \wedge g_4 + d\{F(g_1 \wedge g_3 + g_2 \wedge g_4)\}\right] \\
B &= \frac{g_s M}{2}\left\{f g_1 \wedge g_2 + k g_3 \wedge g_4\right]
\end{aligned} \tag{A.33}$$

where the one-forms g_i are defined by

$$\begin{aligned}
g_1 &= \frac{1}{\sqrt{2}}(-s_1 d\phi_1 - c_\psi s_2 d\phi_2 + s_\psi d\theta_2)\,, \qquad g_2 = \frac{1}{\sqrt{2}}(d\theta_1 - s_\psi s_2 d\phi_2 - c_\psi d\theta_2)\,, \\
g_3 &= \frac{1}{\sqrt{2}}(-s_1 d\phi_1 + c_\psi s_2 d\phi_2 - s_\psi d\theta_2)\,, \qquad g_4 = \frac{1}{\sqrt{2}}(d\theta_1 + s_\psi s_2 d\phi_2 + c_\psi d\theta_2)\,, \\
g_5 &= d\psi + c_1 d\phi_1 + c_2 d\phi_2\,.
\end{aligned} \tag{A.34}$$

We can now use the procedure outlined above to construct the deformed solution. For example, performing $\mathrm{SL}(3,\mathbb{R})$ transformation with parameter γ, we find ingredients of the modified metric

$$B_{12} = \frac{g_s M}{4}(f - k) s_1 s_2 s_\psi\,, \tag{A.35}$$

Then we find

$$\begin{aligned}
(A^1_\mu)' &= (1 + \gamma B_{12}) A^1_\mu + \frac{\gamma g_s M}{4} s_2[(f + k) d\theta_2 + c_\psi (k - f) d\theta_1] \\
(A^2_\mu)' &= (1 + \gamma B_{12}) A^2_\mu + \frac{\gamma g_s M}{4} s_1[(f + k) d\theta_1 + c_\psi (k - f) d\theta_2] \\
G^{-1} &= (1 + \gamma B_{12})^2 + \gamma^2 \det g\,.
\end{aligned} \tag{A.36}$$

Here $\sqrt{\det g}$ is a volume of the 2-torus, and we will not give the explicit forms for A^1, A^2. The important point we want to convey is that the original B field will appear in the deformed metric. The transformed metric can be written in a form

$$\begin{aligned}
ds^2 &= h^{-1/2} m^2 dx_m dx_m + \frac{h^{1/2} d\tau^2}{3^{2/3} 2^{4/3} K^2} + \qquad \text{(A.37)} \\
&\quad + h^{1/2} \frac{3^{1/3}}{2^{4/3}} K G \left[\frac{(g_5)^2}{3K^3} + \cosh^2\frac{\tau}{2}[(g_3)^2 + (g_4)^2] + \sinh^2\frac{\tau}{2}[(g_1)^2 + (g_2)^2] \right] +
\end{aligned}$$

[25] We use the same notation as [23], reader should consult that paper for the explicit form of f, k, h, K. Notice however that to have the same units of flux as we use in the rest of the paper, we redefined a parameter M compared to [23]: $M_{our} = \frac{1}{2} M_{KS}$. Our normalization is also the one used in [19].

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$$+h^{1/2}(1-G)\frac{3^{1/3}}{2^{4/3}}K\left\{\frac{s_1^2 s_2^2\sinh^2\tau}{2H}(\cosh\tau(d\theta_1^2+d\theta_2^2)+2c_\psi d\theta_1 d\theta_2)+\right.$$
$$+\frac{H}{12K^3\mathcal{F}}\left[d\psi-\frac{s_\psi s_1 s_2}{H}(c_1c_\psi s_2+c_2s_1\cosh\tau)d\theta_1-\right.$$
$$\left.\left.-\frac{s_\psi s_1 s_2}{H}(c_2c_\psi s_1+c_1s_2\cosh\tau)d\theta_2\right]^2\right\}.$$

Here we introduced various functions

$$\begin{aligned}
H &\equiv s_1^2 s_2^2(\cosh^2\tau-c_\psi^2)\\
\mathcal{F} &\equiv \frac{1}{6K^3}\left[(c_2^2s_1^2+c_1^2s_2^2)\cosh\tau+2s_1s_2c_1c_2c_\psi\right]+\frac{1}{4}s_1^2s_2^2(\cosh^2\tau-\cos^2\psi)\,,\\
G^{-1} &\equiv \left(1+\frac{\gamma g_s M}{4}(f-k)s_1s_2s_\psi\right)^2+\gamma^2 h\frac{3^{2/3}}{2^{8/3}}K^2\mathcal{F}\,.
\end{aligned}$$

We first look at large values of τ. At leading order we get

$$G^{-1}\sim 1+\gamma^2\left(\frac{3}{2}\right)^{2/3}\frac{\alpha\tau}{64}(c_2^2s_1^2+c_1^2s_2^2+\frac{3}{2}s_1^2s_2^2) \tag{A.38}$$

and the metric becomes

$$\begin{aligned}
ds^2 &= h^{-1/2}m^2dx_mdx_m+\frac{h^{1/2}d\tau^2}{3^{2/3}2^{4/3}K^2}+\\
&+h^{1/2}e^\tau\frac{3^{1/3}}{2^{4/3}}KG\left[\frac{1}{6}(g_5)^2+\frac{1}{4}[(g_1)^2+(g_2)^2+(g_3)^2+(g_4)^2]+\right.\\
&\left.+\gamma^2\left(\frac{3}{2}\right)^{2/3}\frac{\alpha\tau}{256}\left\{\left(c_2^2s_1^2+c_1^2s_2^2+\frac{3}{2}s_1^2s_2^2\right)(d\theta_1^2+d\theta_2^2)+s_1^2s_2^2d\psi^2\right\}\right].
\end{aligned}$$

This metric becomes highly curved for large τ. Notice that in contrast to the conformal case, the γ-deformation grows with τ. Note also that B_{12} decreases for large τ.

At $\tau=0$ we get the approximate expressions

$$\begin{aligned}
K &= (2/3)^{1/3}\,, \qquad H=s_1^2s_2^2s_\psi^2\,,\\
\mathcal{F} &= \frac{1}{4}(c_2^2s_1^2+c_1^2s_2^2+2s_1s_2c_1c_2c_\psi+s_1^2s_2^2s_\psi^2) \qquad G^{-1}=1+\frac{\gamma^2 h}{4}\mathcal{F}
\end{aligned} \tag{A.39}$$

and the metric becomes

$$\begin{aligned}
ds^2 &= h^{-1/2}m^2dx_mdx_m+\frac{h^{1/2}d\tau^2}{3^{2/3}2^{4/3}K^2}+\\
&+\frac{h^{1/2}G}{2}\left[\frac{1}{2}(g_5)^2+(g_3)^2+(g_4)^2+\frac{\gamma^2h}{32}\left(\frac{1}{\sqrt{2}}g_5+c_1s_2s_\psi g_3-(c_2s_1+c_1s_2c_\psi)g_4\right)^2\right].
\end{aligned} \tag{A.40}$$

We see that sphere S^3 which is located at the origin of τ is deformed. It can be checked that G in (A.39) is a function of only the 3-sphere coordinates. Similarly one can check that the last term in (A.40) depends only on the angles on S^3. In other words the metric on the three sphere is deformed as it would be by doing the $SL(2,\mathbb{R})$ transformation on two commuting isometries. We can define new coordinates $\tilde\phi_1, \tilde\theta, \tilde\phi_2$ through [57]

$$e^{i\sigma^3\frac{\tilde\phi_1}{2}}e^{i\sigma^2\frac{\tilde\theta}{2}}e^{i\sigma^3\frac{\tilde\phi_2}{2}} = e^{i\sigma^3\frac{\phi_1}{2}}e^{i\sigma^2\frac{\theta_1}{2}}e^{i\sigma^3\frac{\psi}{2}}e^{i\sigma^2\frac{\tilde\theta_2}{2}}e^{i\sigma^3\frac{\phi_2}{2}}\,. \tag{A.41}$$

Note that under shifts of ϕ_i, the $\tilde\phi_i$ shift in the same way. Then we can write the metric (A.40) as

$$ds^2 = h^{-1/2}m^2 dx_m dx_m + \frac{h^{1/2}d\tau^2}{3^{2/3}2^{4/3}K^2} + \tag{A.42}$$

$$+\frac{h^{1/2}}{4}\left\{G\left[(d\tilde\phi_1 + \cos\tilde\theta d\tilde\phi_2)^2 + \sin^2\tilde\theta d\tilde\phi_2^2\right] + d\tilde\theta^2\right\} \tag{A.43}$$

$$G^{-1} = 1 + \gamma^2\frac{h}{4}\sin^2\tilde\theta\,. \tag{A.44}$$

Notice that only the γ transformation leads to a non-singular metric. If we tried to do a σ transformation on this three sphere we would run into trouble since there is H_{RR} flux on it and $C^{(2)}_{12}$ would not be well defined.

A.4 Marginal deformation of $AdS_4 \times S^7$.

So far we have generated several solutions of type-IIB supergravity using the lift to eleven dimensions and $SL(3,\mathbb{R})$ group of eleven dimensional supergravity on the three-torus. One can also produce new solutions of eleven dimensional supergravity by reducing to type-IIB and using $SL(2,\mathbb{R})$ symmetry there. To illustrate this procedure we consider the example of $AdS_4 \times S^7$

$$ds^2 = \frac{1}{4}ds^2_{AdS} + d\Omega_7^2\,, \qquad F_{(4)} = \frac{3}{2^3}\omega_{AdS_4}\,. \tag{A.45}$$

Let us parameterize the sphere as

$$d\Omega_7^2 = d\theta^2 + s_\theta^2(d\alpha^2 + s_\alpha^2 d\beta^2) + c_\theta^2 d\phi_1^2 + s_\theta^2\left[c_\alpha^2 d\phi_2^2 + s_\alpha^2(c_\beta^2 d\phi_3^2 + s_\beta^2 d\phi_4^2)\right] \tag{A.46}$$

and introduce new angles

$$\phi_1 = \psi + \varphi_3\,, \qquad \phi_2 = \psi - \varphi_3 - \varphi_2\,, \qquad \phi_3 = \psi + \varphi_2 - \varphi_1\,, \qquad \phi_4 = \psi + \varphi_1\,. \tag{A.47}$$

Then we can rewrite the geometry in a form (A.1) with following ingredients:

$$\Delta^{1/3}e^{4\phi/3} = c_\theta^2 + s_\theta^2 c_\alpha^2\,, \qquad \Delta = s_\theta^4 s_\alpha^2(c_\theta^2 c_\alpha^2 + s_\alpha^2 s_\beta^2 c_\beta^2(c_\theta^2 + s_\theta^2 c_\alpha^2)) = \mu_1^2\mu_2^2\mu_3^2\mu_4^2\sum_1^4 \mu_i^{-2}\,,$$

$$\Delta^{1/3}h_{mn}D\varphi^m D\varphi^n = e^{2\phi/3}\left[s_\theta^2 s_\alpha^2(D\varphi_1 - c_\beta^2 D\varphi_2)^2 + s_\theta^2\frac{c_\theta^2 c_\alpha^2 + s_\alpha^2 s_\beta^2 c_\beta^2(c_\theta^2 + s_\theta^2 c_\alpha^2)}{c_\theta^2 + s_\theta^2 c_\alpha^2}D\varphi_2^2\right]$$

$$\mathcal{A}^1 = \frac{-4(1+2c_{2\beta})c_\theta^2 c_\alpha^2 + s_\alpha^2 s_{2\beta}^2(c_\theta^2 + s_\theta^2 c_\alpha^2)}{4c_\theta^2 c_\alpha^2 + s_\alpha^2 s_{2\beta}^2(c_\theta^2 + s_\theta^2 c_\alpha^2)}d\psi\,, \quad \mathcal{A}^2 = 2\frac{-4c_\theta^2 c_\alpha^2 + s_\alpha^2 s_{2\beta}^2(c_\theta^2 + s_\theta^2 c_\alpha^2)}{4c_\theta^2 c_\alpha^2 + s_\alpha^2 s_{2\beta}^2(c_\theta^2 + s_\theta^2 c_\alpha^2)}d\psi\,,$$

$$N_1 = 0\,, \qquad N_2 = \frac{s_\theta^2 c_\alpha^2}{c_\theta^2 + s_\theta^2 c_\alpha^2}\,, \qquad \mathcal{A}^3 = \left(1 - \frac{4s_\alpha^2 s_{2\beta}^2 s_\theta^2 c_\alpha^2}{4c_\theta^2 c_\alpha^2 + s_\alpha^2 s_{2\beta}^2(c_\theta^2 + s_\theta^2 c_\alpha^2)}\right)d\psi$$

$$\Delta^{-1/6}g_{\mu\nu}dx^\mu dx^\nu = \frac{1}{4}ds^2_{AdS} + d\theta^2 + s_\theta^2(d\alpha^2 + s_\alpha^2 d\beta^2) + \frac{s_{2\theta}^2 s_{2\alpha}^2 s_{2\beta}^2 d\psi^2}{4c_\theta^2 c_\alpha^2 + s_\alpha^2 s_{2\beta}^2(c_\theta^2 + s_\theta^2 c_\alpha^2)}\,.$$

The geometry of type-IIB becomes:

$$\begin{aligned}
ds^2_{IIB} &= \frac{1}{h_{11}}\frac{(d\varphi_1)^2}{\sqrt{\Delta}} + \frac{\sqrt{\Delta}}{h_{11}}(D\varphi^2)^2 + e^{2\phi/3}g_{\mu\nu}dx^\mu dx^\nu\,,\\
B &= -c_\beta^2 d\varphi^1\wedge D\varphi^2 + d\varphi^1\wedge\mathcal{A}^1\\
e^{2\Phi} &= \frac{e^{2\phi}}{h_{11}}\,,\qquad C^{(0)}=0\\
C^{(2)} &= -\frac{s_\theta^2 c_\alpha^2}{c_\theta^2+s_\theta^2c_\alpha^2}d\varphi^1\wedge D\varphi^2 - d\varphi^1\wedge\mathcal{A}^3\,,\\
C^{(4)} &= -\frac{3}{8}\left(w_3 d\varphi^1 + \sqrt{\Delta}\hat{w}_3 d\varphi^2\right),\qquad dw_3=\omega_{AdS_4}\,,\qquad d\hat{w}_3=\sqrt{\Delta}\,{}^*_8 dw_3\,.
\end{aligned}\tag{A.48}$$

Making $\mathrm{SL}(2,\mathbb{R})$ transformation in the φ^1-φ^2 plane:

$$\begin{pmatrix}\varphi^1\\ \varphi^2\end{pmatrix}\to\begin{pmatrix}\hat\alpha & \hat\beta\\ \hat\gamma & \hat\delta\end{pmatrix}\begin{pmatrix}\varphi^1\\ \varphi^2\end{pmatrix}\tag{A.49}$$

we find a more general geometry and we can read off the modified ingredients of eleven dimensional metric. First we look at

$$\Delta'=\frac{\Delta}{(\hat\alpha^2+\hat\gamma^2\Delta)^2}\,,\qquad h'_{11}=h_{11}\,,\quad C'_{123}=-\frac{\hat\alpha\hat\beta+\hat\gamma\hat\delta\Delta}{\hat\alpha^2+\hat\gamma^2\Delta}\,.\tag{A.50}$$

The metric should be regular as $\Delta\to 0$, this selects the value $\hat\alpha=1$. Also at $\Delta=0$ the 2-torus contracts, so the components of tensor fields along this torus should go to zero. In particular, C'_{123} should go to zero at these points which leads to $\hat\beta=0$. Setting $\hat\alpha=\hat\delta=1$, $\hat\beta=0$, we get the expressions for the transformed quantities:

$$\begin{aligned}
\Delta' &= \frac{\Delta}{(1+\hat\gamma^2\Delta)^2}\,,\quad h'_{11}=h_{11}\,,\quad e^{2\phi'}=e^{2\phi}\,,\quad g'_{\mu\nu}=g_{\mu\nu}\,,\\
(\mathcal{A}^a)' &= \mathcal{A}^a\,,\quad N'_m=N_m\,,\quad C'_{123}=-\frac{\hat\gamma\Delta}{1+\hat\gamma^2\Delta}\,,\quad C'_{12\mu}=0\,,\\
C'_{1\mu\nu} &= C'_{2\mu\nu}=0\,,\qquad C'_{1\mu}=0\,,\quad C'_{\mu\nu}=0\,.
\end{aligned}\tag{A.51}$$

Substituting this into the eleven dimensional geometry, we get

$$\begin{aligned}
ds^2_{11} &= G^{-1/3}\left[\frac{1}{4}ds^2_{AdS}+\sum(d\mu_i^2+G\mu_i^2d\phi_i^2)+16G\hat\gamma^2\mu_1^2\mu_2^2\mu_3^2\mu_4^2(\sum d\phi_i)^2\right]\\
F_{(4)} &= \frac{3}{8}(\omega_{AdS}+16\hat\gamma s_\theta^5 c_\theta s_{2\alpha}^2 s_{2\beta}d\theta d\alpha d\beta d\psi)-\hat\gamma d\left\{\Delta G D\varphi_1 D\varphi_2 D\varphi_3\right\}
\end{aligned}\tag{A.52}$$

$$\Delta=\mu_1^2\mu_2^2\mu_3^2\mu_4^2\sum_1^4\mu_i^{-2}\,,\qquad G^{-1}=1+\hat\gamma^2\Delta\,.\tag{A.53}$$

Notice that

$$D\varphi_1=d\varphi_1+d\psi-\frac{s_\theta^4 s_\alpha^2}{\Delta}4c_\beta^2c_\theta^2c_\alpha^2 d\psi=d\phi_4-\frac{4\prod\mu_i^2}{\Delta}\frac{d\psi}{\mu_4^2}\,,\tag{A.54}$$

$$D\varphi_2=d\varphi_2+2d\psi-\frac{4}{\Delta}s_\theta^4s_\alpha^2c_\theta^2c_\alpha^2d\psi=d\phi_3+d\phi_4-\frac{4\prod\mu_i^2}{\Delta}\left(\frac{d\psi}{\mu_3^2}+\frac{d\psi}{\mu_4^2}\right),\tag{A.55}$$

$$D\varphi_3=d\varphi_3+d\psi-\frac{1}{\Delta}s_\theta^4s_\alpha^4s_{2\beta}^2s_\theta^2c_\alpha^2d\psi=d\phi_1-\frac{4\prod\mu_i^2}{\Delta}\frac{d\psi}{\mu_1^2}\,.\tag{A.56}$$

Using these expressions, we can simplify the deformed solution in (5.3).

B. Classical solutions for the probe string

Lets us analyze the motion of a probe string in the geometry (3.14). We look for a solution of the form

$$\psi = \psi_1(\tau) + \psi_2(\sigma)\,, \quad \varphi_1(\sigma)\,, \quad \varphi_2(\sigma)\,, \quad t = ER^{-2}\tau \tag{B.1}$$

while coordinates α and θ are constant. With this ansatz we have to impose the constraints

$$g_{\psi\mu}(x^\mu)' = 0\,, \qquad R^{-2}E^2 = g_{\psi\psi}\dot{\psi}^2 + g_{\mu\nu}(x^\mu)'(x^\nu)'\,. \tag{B.2}$$

Then we find

$$\begin{aligned} G^{-1}R^{-4}(E^2-J^2) &= s_\alpha^2(4-(12-8c_{2\theta}^2)s_\alpha^2+9s_{2\theta}^2 s_\alpha^4)\times \\ &\times\left\{(1+9\hat{\gamma}^2 s_\theta^2 c_\theta^2 s_\alpha^4 c_\alpha^2)A^2 + \frac{s_{2\theta}^2[1+\hat{\gamma}^2 s_\alpha^2(c_\alpha^2+s_\alpha^2 s_\theta^2 c_\theta^2)]^2}{4c_{2\theta}^2 c_\alpha^2(1+9\hat{\gamma}^2 s_\theta^2 c_\theta^2 s_\alpha^4 c_\alpha^2)}B^2\right\}. \end{aligned} \tag{B.3}$$

We have introduced coefficients A and B which are related to various quantities in the following way

$$\begin{aligned} \psi' &= -\frac{4c_\alpha^2 - 8c_{2\theta}^2 s_\alpha^2 + 9s_{2\theta}^2 s_\alpha^4}{6c_{2\theta}c_\alpha^2(1+9\hat{\gamma}^2 s_\theta^2 c_\theta^2 s_\alpha^4 c_\alpha^2)}B\,, \\ \varphi_1' &= 2c_{2\theta}s_\alpha^2 A - c_{2\theta}s_\alpha^2\hat{\gamma}\dot{\psi} - \frac{c_{2\theta}^2 s_\alpha^2 + 2 - 3s_\alpha^2}{3c_{2\theta}c_\alpha^2}B \\ \varphi_2' &= (c_\alpha^2 - s_\alpha^2 c_\theta^2)(2A - \hat{\gamma}\dot{\psi}) + B\left[1 + \frac{c_{2\theta}^2 s_\alpha^2 + 2 - 3s_\alpha^2}{6c_{2\theta}c_\alpha^2}\right] \\ R^{-2}J &= \dot{\psi} - \frac{\hat{\gamma}}{2}s_\alpha^2 AG(4-(12-8c_{2\theta}^2)s_\alpha^2 + 9s_{2\theta}^2 s_\alpha^4)\,. \end{aligned} \tag{B.4}$$

The advantage of writing (B.3) in terms of A and B is that the expression (B.3) is positive definite and for generic values of α, θ the BPS condition leads to $A = B = 0$ which translates into

$$\psi' = 0\,, \qquad \varphi_1' = -c_{2\theta}s_\alpha^2\hat{\gamma}\dot{\psi}\,, \qquad \varphi_2' = -\hat{\gamma}(c_\alpha^2 - s_\alpha^2 c_\theta^2)\dot{\psi}\,. \tag{B.5}$$

One can also check that equations of motion for α and θ are satisfied. Let us look at the angular momenta corresponding to the solution (B.5)

$$J_{\varphi_1} = -(c_\alpha^2 - s_\alpha^2 c_{2\theta})J = R^2\frac{\varphi_2'}{\hat{\gamma}}\,, \quad J_{\varphi_2} = s_\alpha^2 c_{2\theta}J = -R^2\frac{\varphi_1'}{\hat{\gamma}}\,, \qquad J_\psi = R^2\dot{\psi} \tag{B.6}$$

For closed string we need that

$$\varphi_1' = n_1\,, \quad \varphi_2' = n_2\,, \qquad n_{1,2} \in Z \tag{B.7}$$

since at generic values of $\alpha,\ \theta$ the two circles have a non-vanishing size. In terms of these integers we find

$$J_{\varphi_1} = \frac{n_2}{\gamma}\,, \quad J_{\varphi_2} = -\frac{n_1}{\gamma}\,.$$

These can only be integers if γ is a rational number. When $\gamma = m/n$, then we see that the momenta J_{φ_i} are multiples of n. We will also need the expressions for three angular momenta (J_1, J_2, J_3) which are related to the quantities introduced above

$$J = J_1 + J_2 + J_3\,, \qquad J_{\varphi_1} = J_2 - J_1\,, \qquad J_{\varphi_2} = J_2 - J_3\,. \tag{B.8}$$

So finally we find the expressions

$$J_3 = J_2 + \frac{n_1}{\gamma}\,, \qquad J_2 = J_1 + \frac{n_2}{\gamma}\,. \tag{B.9}$$

Note also that the relation (B.6) between the values of α, θ and the J_i is such that

$$(|J_1|, |J_2|, |J_3|) = \lambda(\mu_1^2, \mu_2^2, \mu_3^2) \tag{B.10}$$

for some λ.

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